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Robust nonlinear Model Predictive Control based on nominal predictions: A zonotopic approach

Victor Moreira Cunha

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Resumo

O principal objetivo deste trabalho é o desenvolvimento, análise e simulação de novos algoritmos de controle preditivo robusto para sistemas não-lineares na presença de perturbações aditivas limitadas. Os controladores propostos satisfazem as propriedades de factibiildade recursiva e estabilidade entrada-estado e foram desenvolvidos tomando como base algoritmos existentes para os casos nominal ou linear. Satisfação robusta das restrições é garantida através de predições nominais e restrições contraídas, com a extensão de valor médio de zonotopos sendo utilizada de modo a reduzir o conservadorismo na propagação de incertezas.

Os problemas de regulação sem *offset* na presença de perturbações constantes e seguimento de referências constantes por partes foram estudados, considerando também perturbações estocásticas e restrições de estado probabilísticas. Finalmente, as estratégias propostas foram aplicadas em simulação aos sistemas de referência *Buck-Boost* e CSTR (*Continually Stirred Tank Reactor*), de modo a validar os controladores e ilustrar suas propriedades.

Palavras-Chave: *Controle Preditivo, Controle não-linear, Controle Robusto, Restrições Contraídas, Conjuntos Invariantes, Zonotopos.*

Abstract

The main objective of this dissertation is the development, analysis and simulation of new robust model predictive control algorithms for nonlinear systems in the presence of bounded additive disturbances. The proposed controllers satisfy recursive feasibility and input-to-state stability criteria. They are initially derived from existing algorithms for nominal or linear models. Robust constraint satisfaction is reached through nominal predictions coupled with tightened constraints, with the mean-value zonotopic extension being used in order to reduce conservatism in the disturbance propagation.

The problems of regulation without offset in the presence of constant disturbances and tracking of piece-wise constant references were tackled, also considering stochastic disturbance and chance state constraints. The proposed techniques are applied to the Buck-Boost and CSTR (Continually Stirred Tank Reactor) simulation case studies in order to validate and illustrate the proposed approaches.

Keywords: *Model Predictive Control, Nonlinear Control, Robust Control, Constraint Tightening, Invariant Sets, Zonotopes.*

Notation

Given the sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^m$, $\mathcal{C} \subseteq \mathbb{R}^n$ and the matrix $R \in \mathbb{R}^{n\times m}$, the Minkowski sum is defined as $\mathcal{A} \oplus \mathcal{B} = \{x \in \mathbb{R}^m \colon x = a + b,\ a \in \mathcal{A},\, b \in \mathcal{B}\}$, the Pontryagin difference as $\mathcal{A} \ominus \mathcal{B} = \{x \in \mathbb{R}^m \colon x + b \in \mathcal{A},\ \forall b \in \mathcal{B}\}$, the linear mapping as $R\mathcal{A} = \{y \in \mathbb{R}^n \colon y = Ra,\ a \in \mathcal{A}\}$ and the cartesian product as $\mathcal{A} \times \mathcal{C} = \{z \in \mathbb{R}^{m+n} \colon z = (a, c),\ a \in \mathcal{A},\, c \in \mathcal{C}\}$.

The term x_k represents the value of a signal on instant k, while $\Delta x_k = x_k - x_{k-1}$ represents its first difference, and $x_{k+j|k}$ is the value of x_{k+j} as predicted in k (note that $x_{k|k} = x_k$). Given two integers a and b, with $a \leq b$, and a signal v_k, then the set of integers between a and b is described by $\mathbb{Z}_{[a,b]} = \{j \in \mathbb{Z} \colon a \leq j \leq b\}$, and the sequence defined by v_k with k between a and b is represented by $\mathbf{v}_{[a,b]} = \{v_a, v_{a+1}, \ldots, v_b\}$.

Given the matrices $A, B \in \mathbb{R}^{n\times m}$, $A \leq (\geq) B$ represents the mn inequations $a_{ij} \leq (\geq) b_{ij}$, while $A \preceq (\succeq) B$ means that $A - B$ is a negative (positive) semidefinite matrix. A (unspecified) norm of a vector $v \in \mathbb{R}^n$ is represented by $\|v\|$, while $\|v\|_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}$ is its p-norm and $\|v\|_\infty = \max_{i\in\mathbb{Z}_{[1,n]}} |v_i|$ its infinity-norm. For a given matrix $A \in \mathbb{R}^{n\times m}$, $\|A\|$ ($\|A\|_p$, $\|A\|_\infty$) is the induced (p-, infinity-)norm of the linear transformation $A \colon \mathbb{R}^m \to \mathbb{R}^n$. The absolute value $|A|$ of a matrix must be taken term-by-term. The notation $\|\mathbf{w}_{[0,j]}\|$ represents the norm of a sequence $\mathbf{w}_{[0,j]} = \{w_0,\ \ldots,\ w_j\}$, i.e. $\|\mathbf{w}_{[a,b]}\| = \max_{j\in\mathbb{Z}_{[a,b]}} \|w_j\|$.

A function $\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\alpha(0) = 0$, and it is a $\mathcal{K}_\infty$-function if $\lim_{s\to\infty} \alpha(s) = \infty$ as well. A function $\beta \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{KL}$-function if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is a $\mathcal{K}$-function and, for each $s \geq 0$, $\beta(s, \cdot)$ is decreasing, with $\lim_{t\to\infty} \beta(s,t) = 0$. A function $f \colon \mathcal{A} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is said to be Lipschitz continuous if there exists a constant $L \in \mathbb{R}$ such that $\|f(x_b) - f(x_a)\| \leq L\|x_b - x_a\|$, $\forall x_a, x_b \in \mathcal{A}$ and of class $\mathcal{C}^1$ if it is differentiable and has continuous first-order derivatives. In this case, its jacobian matrix is represented by $\nabla^\intercal f \colon \mathcal{A} \to \mathbb{R}^{n\times m}$.

The unitary m-dimensional box is described by $\mathcal{B}_\infty^m = \{\xi \in \mathbb{R}^m \colon \|\xi\|_\infty \leq 1\}$, and the set of real compact intervals is given by $\mathbb{I} = \{[a, b],\ a, b \in \mathbb{R},\, a \leq b\}$. Given a set $\mathcal{A} \subseteq \mathbb{R}^m$, $\mathbb{I}(\mathcal{A}) \in \mathbb{I}^m$ represents its interval hull. Interval matrices are represented by $\mathbf{J} \in \mathbb{I}^{n\times m}$, with $mid(\mathbf{J})$ and $rad(\mathbf{J})$ representing, respectively, its medium point and radius.

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Chapter 1

Introduction

1.1 Motivation

Model Predictive Control (MPC) [16] presents an alternative approach to classical control strategies, allowing for a formal incorporation of constraints via the MPC optimization problem, which integrates performance and constraint satisfaction. The receding horizon paradigm states that the optimization problem must be solved at each time instant, so as to obtain the next control action.

Earlier MPC strategies, however, did not ensure recursive feasibility or stability. Therefore, relatively large prediction horizons, associated with high online computational costs (specially for nonlinear systems), should be employed so as to avoid constraint violation or even instability. Strategies like the introduction of terminal positively invariant sets and terminal costs [25] were then considered in order to ensure recursive feasibility of the optimization problem and asymptotical stability of the closed-loop system, even for shorter horizons.

However, even with formal guarantees of stability and constraint satisfaction for the nominal system, applied in the prediction, the presence of modelling errors and disturbances can still deteriorate the controller performance, or even result in instability. In order to allow for such guarantees to hold under the presence of uncertainties, robust model predictive control has been developed. Robust MPC considers the presence of unknown but bounded uncertainties and ensures robust constraint satisfaction and Input-to-State (ISS) stability [12] for any admissible sequence of disturbances [27, 23].

Robust Nonlinear Model Predictive Control thus consists of an active area of research, with recent works [34, 13] applying nominal predictions, tightened constraints and terminal robust positively invariant sets in order to ensure robust stability. Nonetheless, related problems, such as: (i) the search for less conservative tightened constraints and terminal sets, (ii) offset correction in the presence of constant disturbances, and (iii) the extension of these results for the tracking case, are still in development and are the focus of this work.

1.2 Model Predictive Control

Model Predictive Control is a control strategy that uses a model to predict the future behavior of the system and find, via optimization, the 'best' sequence of future inputs for the prescribed finite horizon problem, i.e. the one that minimizes a given cost function, subject to certain state and input constraints. The control horizon represents how many future inputs are considered as variables of the MPC optimization problem and the prediction horizon defines how many future states are considered in the prediction. Following the receding horizon paradigm, only the first input in the optimal control sequence calculated is applied, and the optimization problem is solved again in the next time instant, thus closing the loop.

The first proposed MPC strategies (DMC (Dynamic Matrix Control), GPC (Generalized Predictive Control), and State-Space MPC) applied models based on the step response, transfer function and state-space to predict and control the behavior of linear systems [16]. Quadratic cost functions and linear constraints were also considered, resulting in quadratic programming problems. These control strategies allow the natural incorporation of constraints, with the State-Space MPC also being easily extended to the nonlinear case, albeit with a more complex, in general non-convex, associated optimization problem, and thus, higher computational costs.

However, with the prediction horizon N being finite, information about the behavior of the system after the time instant $k+N$ is not considered at k. Therefore, recursive feasibility and stability are not in general guaranteed. A strategy to avoid feasibility loss and instability is the application of a 'sufficiently large' prediction horizon, involving virtually all the prediction dynamic. This, however, is associated to a high, sometimes unviable, computational cost, specially for nonlinear dynamics.

In this context, methods for guaranteeing stability of MPC strategies were proposed. In particular, a terminal positively invariant set and a terminal cost function can be incorporated in order to ensure recursive feasibility and stability of the closed-loop system [27, 16]. These guarantees, however, consider that the process evolution will be identical to the nominal, predicted, one, and thus are invalid in the presence of prediction errors or additive disturbances.

A strategy to deal with the presence of uncertainties is considering nominal predictions and tightened constraints in the MPC optimization problem, such that the real system trajectory, in the presence of uncertainties, satisfies the original constraints [19, 34]. Terminal robust positively invariant sets and terminal costs are also employed. This way, it is possible to ensure recursive feasibility and Input-to-State stability even in the presence of additive, limited, uncertainties.

For computation of the tighter constraints, disturbance propagation sets, which limit the difference between predictions made at k and $k+1$, are necessary. For linear systems, these sets can be directly obtained from the model matrices. For nonlinear dynamics, however, the existing methods of disturbance propagation, such as the one based on Lipschitz constants, tend to be rather conservative. In this context, zonotopes appear as an interesting alternative. They are a class of convex, symmetric polytopes that, due to their flexibility and simplicity, allied with the low computational cost of their linear transformations and Minkowski sums, are extensively used in state estimation and fault detection [10, 1, 35].

Finally, the control strategies mentioned above consider the problem of regulation to a given admissible equilibrium, with the prediction model being translated such that this target is represented by the origin. If tracking of piece-wise constant references is required, these controllers could in theory be used, with the equilibrium point being updated at each reference change. The feasibility of the optimization problem can, however, be lost during reference changes, unviabilizing such strategies.

In order to avoid feasibility loss due to reference changes and to increase the domain of attraction, a virtual reference can be included as an additional variable in the MPC optimization problem [21, 22]. The freedom provided by the artificial reference permits recursive feasibility and stability guarantees for the tracking problem, with the artificial reference tending to the real one via the incorporation of an offset cost.

1.3 Main Contributions

The main contributions of this project are detailed as follows:

(i) Proposal of a disturbance propagation method based on zonotopes, which is shown to be less conservative than the typical solution, and thus results in less tightened constraints than existing strategies.

(ii) Incorporation of mean-value disturbance estimations into the model prediction and target correction in order to avoid steady-state offset in the presence of constant disturbances.

(iii) Extension of the nominal tracking NMPC proposed in [22] to the robust case, ensuring recursive feasibility and ISS-stability of the ensuing controller.

(iv) Incorporation of chance state constraints [34] into the proposed controllers, maintaining feasibility and stability guarantees.

Other contributions include the development of an algorithm for computing the closed-loop prediction matrix in order to mitigate the disturbance propagation, a terminal constraint contraction approach which generalizes the method of calculating Robust Positively Invariant (RPI) sets for nonlinear systems based on linear approximations proposed in [14] and an analysis of the properties of the zonotopic mean-value extension [1].

1.4 Related Papers

In this section, related papers, which were made during this masters and on which most of this dissertation is based, are briefly presented.

1. The paper *Controle Preditivo não-linear Robusto com Propagação de Incertezas via Zonotopos* [6] was presented in the Congresso Brasileiro de Automática (CBA) 2020, where the zonotopic disturbance propagation method was proposed, applied in the design of a regulating NMPC for the Buck-Boost DC-DC converter and shown to be less conservative than the infinity-norm Lipschitz method.

2. The paper *Robust Nonlinear Model Predictive Control with Bounded Disturbances based on Zonotopic Constraint Tightening* [8], published in the Journal of Control, Automation and Electric Systems (JCAE), is an extended version of the previous paper, presenting the constant disturbance attenuation method and a new case study, based on the Continually Stirred Tank Reactor (CSTR).

3. The paper *Robust Nonlinear Model Predictive Control based on nominal predictions with piecewise constant references and bounded disturbances* [7], published in the International Journal of Robust and Nonlinear Control (IJRNC), proposes the robust tracking NMPC strategy, with the introduction of an artificial reference in order to avoid feasibility loss due to reference changes [22]. In this paper, stochastic disturbances and chance constraints were also considered.

Furthermore, the papers *Robust MPC for linear systems with bounded disturbances based on admissible equilibria sets* [33] and *Robust Nonlinear Predictive Control through qLPV embedding and Zonotope Uncertainty Propagation* [29], although not directly presented in this dissertation, are also related to this project. In [33], the robust MPC of linear systems is considered, applying terminal equality constraints and maintaining recursive feasibility through the introduction of appropriate *slacks*, while in [29] the zonotopic disturbance propagation method proposed is applied to quasi-Linear Parameter Varying (qLPV) systems.

1.5 Structure of the Text

This dissertation is structured as follows: Chapter 2 presents the robust NMPC through constraint tightening on which this work is based, as well as other fundamentals, such as the disturbance propagation process, robust positively invariant sets and zonotopes. In Chapter 3, a constant disturbance model is included into the NMPC prediction, and target correction is applied in order to avoid offset in the presence of constant disturbances. Chapter 4 presents the zonotopic disturbance propagation approach and its associated conservatism reduction, while Chapter 5 considers the robust tracking of piecewise constant references, with recursive feasibility and stability guarantees through the inclusion of an artificial reference. Chapter 6 considers the presence of stochastic disturbances and chance constraints. Finally, simulation case-studies are presented in Chapter 7, and concluding remarks are presented in Chapter 8.

Chapter 2

Robust NMPC based on Nominal Predictions

In this chapter, the basic aspects and properties of the robust nonlinear model predictive control based on nominal predictions are presented. First, the general state-space model of discrete-time nonlinear systems with additive uncertainties and state and input constraints is presented. Then, the closed-loop paradigm, with the introduction of a virtual control in order to mitigate the disturbance propagation through the system dynamics, is considered. Disturbance propagation sets, which are necessary for the constraint tightening and consequent robust constraint satisfaction, are then discussed and the robust NMPC algorithm is presented. Common assumptions which simplify the NMPC design and a method for computing terminal robust positively invariant sets are then introduced. Finally, a brief discussion of zonotopes and associated operations is made.

2.1 System Description

Consider the following discrete-time nonlinear system

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + w_k, \\ y_k &= h(x_k, u_k), \end{aligned} \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ the control input, $y_k \in \mathbb{R}^p$ the controlled output, and $w_k \in \mathbb{R}^n$ the additive disturbance. The functions $f\colon \mathbb{R}^{n\times m} \to \mathbb{R}^n$ and $h\colon \mathbb{R}^{n\times m} \to \mathbb{R}^p$ describe, respectively, the model dynamics and output equations and are considered of class $\mathcal{C}^1$ in the set of admissible states and inputs. It is assumed, without loss of generality, that the origin is an equilibrium point of the system (2.1), such that

$f(0,0)=0$ and $h(0,0)=0$.[1]

Although the additive disturbance is unknown, it is considered to be contained in a compact set, i.e. a bounded and closed set [3, Chapter 5], $\mathcal{W} \subseteq \mathbb{R}^n$ with the origin in its interior, such that $w_k \in \mathcal{W}$, $\forall k \in \mathbb{N}$. System (2.1) is also subject to compact constraints on states and inputs, i.e. there exists a compact set $\mathcal{Z} \subseteq \mathbb{R}^{n+m}$ in such a way that

$$(x_k, u_k) \in \mathcal{Z}, \quad \forall k \in \mathbb{N}. \tag{2.2}$$

Remark 2.1. *For simplicity, $w_k \in \mathcal{W}$ is considered in this work to be an additive disturbance, as in [26] and [5]. However, other sources of uncertainty can be represented by* (2.1) *and thus handled analogously. Considering modeling errors, for instance, an additive uncertainty given by $\tilde{w}_k = \tilde{f}(x_k, u_k) - f(x_k, u_k)$ can be defined, where $\tilde{f}$ and f represent the real system dynamics and prediction model, respectively.*

2.2 Closed-loop Prediction Paradigm

Considering the presence of disturbances, a closed-loop prediction is implemented with the goal of reducing the disturbance propagation through the system dynamics [32, Chapter 7]. The control input is thus given by

$$u_k = \pi(x_k, v_k), \quad \forall k \in \mathbb{N}, \tag{2.3}$$

where x_k is the current available state, and $v_k \in \mathbb{R}^m$ is the virtual input, which satisfies the role of constraint satisfaction and optimization. The feedback function $\pi \colon \mathbb{R}^{n+m} \to \mathbb{R}^m$, that can be chosen so as to mitigate the disturbance propagation, is considered to be of class $\mathcal{C}^1$ and, for each pair $(x,u) \in \mathbb{R}^{n \times m}$, there exists only one $v \in \mathbb{R}^m$ such that $u = \pi(x,v)$. Considering this closed-loop paradigm, the nonlinear model can be alternatively described as

$$\begin{aligned} x_{k+1} &= f(x_k, \pi(x_k, v_k)) + w_k \\ &= f_\pi(x_k, v_k) + w_k, \qquad (2.4a) \\ y_k &= h(x_k, \pi(x_k, v_k)) \\ &= h_\pi(x_k, v_k). \qquad (2.4b) \end{aligned}$$

Moreover, the constraints can be rewritten in terms of the virtual input by the following alternative representation:

$$(x_k, v_k) \in \mathcal{Z}_\pi = \{(x,v) \in \mathbb{R}^{n+m} \colon (x, \pi(x,v)) \in \mathcal{Z}\}, \quad \forall k \in \mathbb{N}. \tag{2.5}$$

[1] If a different equilibrium point is sought, a modified nonlinear model can be defined from translated variables, such that this equilibrium is represented by the origin of the new state-space description.

This description is used to define the NMPC ingredients with respect to the virtual inputs $v_{k+j|k},\ j = 0 \dots N-1$. The sequence of future virtual inputs, namely $\hat{\mathbf{v}}_{[k,k+N-1]} = (v_{k|k}\ v_{k+1|k}\ \dots\ v_{k+N-1|k})$ is then the decision vector for optimization purposes.

2.3 Disturbance Propagation

Consider that the trajectory of system (2.4a), starting from the initial state $x_k \in \mathbb{R}^n$, is given by

$$x_{k+j} = \phi_\pi(j, x_k, \mathbf{v}_{[k,k+j-1]}, \mathbf{w}_{[k,k+j-1]}), \quad j \geq 0. \tag{2.6}$$

It should be remarked that the general analytical expression of the function $\phi_\pi(\cdot)$ is not available, but it can be defined recursively through iterations on $x_{k+j+1} = f_\pi(x_{k+j}, v_{k+j}) + w_{k+j}$ from x_k. Nominal predictions are obtained by considering the null disturbance case, that is

$$x_{k+j|k} = \phi_\pi(j, x_k, \hat{\mathbf{v}}_{[k,k+j-1]}, \mathbf{0}), \quad j \geq 0. \tag{2.7}$$

In order to guarantee recursive feasibility of the NMPC in the presence of disturbances $w_k \in \mathcal{W}$, sets $\mathcal{S}(j),\ j = 0 \dots N$, satisfying Condition 2.1 are iteratively defined so as to limit the disturbance propagation [34, 19].

Condition 2.1. *The disturbance propagation sets $\mathcal{S}(j),\ j = 0 \dots N$, must satisfy the following conditions:*

(i) $\mathcal{S}(0)$ is a compact set that contains $\mathcal{W}$.

(ii) $\mathcal{S}(j),\ j = 1 \dots N$, is a compact set such that, for all $x_a,\ x_b$ and v, with $(x_a, v) \in \mathcal{Z}_\pi \ominus (\mathcal{S}(j-1) \times \{0\})$ and $x_b - x_a \in \mathcal{S}(j-1)$, we have $f_\pi(x_b, v) - f_\pi(x_a, v) \in \mathcal{S}(j)$.

Considering $x_b = x_{k+1}$ and $x_a = x_{k+1|k} = f_\pi(x_k, v_k)$, we have $x_b - x_a \in \mathcal{W} \subseteq \mathcal{S}(0)$. Therefore, from induction on Condition 2.1, $x_{k+j|k+1} \in x_{k+j|k} \oplus \mathcal{S}(j-1),\ j = 1 \dots N+1$, for all admissible control sequences $\hat{\mathbf{v}}_{[k,k+N]}$[2]. The sets $\mathcal{S}(j)$ are thus able to limit the difference between predictions made at the instants k and $k+1$.[3]

In the linear case, the smallest sets $\mathcal{S}^*(j)$ that satisfy Condition 2.1 can be directly computed as $\mathcal{S}^*(j) = (A + BK_v)^j \mathcal{W}$, where $f(x, u) = Ax + Bu$ and $\pi(x, v) = v + K_v x$ [9]. For nonlinear systems, on the other hand, there are no known algorithms for obtaining the optimal $\mathcal{S}^*(j)$ [13]. More conservative outer bounds, taking into account the worst-case disturbance propagation, must then be used.

[2]An admissible future control sequence represents a sequence $\hat{\mathbf{v}}_{[k,k+N]}$ such that $(x_{k+j|k}, v_{k+j|k}) \in \mathcal{Z}_\pi \ominus (\mathcal{S}(j-1) \times \{0\}),\ \forall j = 1 \dots N$.

[3]The sets $\mathcal{S}(j),\ j = 1 \dots N$, presented here and in [34], are related to the disturbance reachable sets $\mathcal{R}(j)$ of [9] in that the $\mathcal{R}(j)$ are given by the accumulation of the disturbance effects $\mathcal{S}(j)$ over the prediction horizon, i.e. $\mathcal{R}(j) = \mathcal{S}(0) \oplus \cdots \oplus \mathcal{S}(j)$.

2.4 Robust NMPC based on Nominal Predictions

Robust Model Predictive Controllers based on nominal predictions apply (2.7) in the formulation of the optimization problem and computation of the optimal sequence of future inputs. However, given the presence of disturbances, tightened constraints are considered in order to avoid constraint violation by the real system trajectories. Terminal cost and constraints are also adapted to the robust case, in order to guarantee recursive feasibility and input-to-state stability [27].

Given the prediction horizon $N \in \mathbb{N}$ and the initial set of constraints $\mathcal{Z}_\pi(0) = \mathcal{Z}_\pi$, tightened constraint sets $\mathcal{Z}_\pi(j),\ j = 1 \dots N$, can be iteratively defined based on the disturbance propagation sets $\mathcal{S}(j)$ as follows

$$\mathcal{Z}_\pi(j+1) = \mathcal{Z}_\pi(j) \ominus (\mathcal{S}(j) \times \{0\}). \tag{2.8}$$

Therefore, at each sampling instant $k \in \mathbb{N}$, the state x_k is obtained and the following optimization problem $P_N(x_k)$ is solved:

$$\min_{\hat{\mathbf{v}}_{[k,k+N-1]}} \sum_{j=0}^{N-1} L_\pi(x_{k+j|k}, v_{k+j|k}) \ + \ V_f(x_{k+N|k}) \tag{2.9a}$$

$$s.t:$$

$$x_{k+j+1|k} = f_\pi(x_{k+j|k}, v_{k+j|k}), \qquad j \in \mathbb{Z}_{[0,N-1]}, \tag{2.9b}$$

$$(x_{k+j|k}, v_{k+j|k}) \in \mathcal{Z}_\pi(j), \qquad j \in \mathbb{Z}_{[0,N-1]}, \tag{2.9c}$$

$$x_{k+N|k} \in \mathcal{X}_f, \tag{2.9d}$$

where $\hat{\mathbf{v}}_{[k,k+N-1]}$ are the future virtual inputs, variables of the optimization problem, $L_\pi(x_{k+j|k}, v_{k+j|k})$ is the stage cost, $V_f(x_{k+N|k})$ is the terminal cost, and $\mathcal{X}_f$ is the terminal set.

Remark 2.2. *In practice, the predicted sequence of states $\hat{\mathbf{x}}_{[k,k+N]}$ is also a variable of the optimization problem. However, this trajectory is fixed by $x_{k|k} = x_k$ and* (2.9b) *for a given sequence of future inputs $\hat{\mathbf{v}}_{[k,k+N-1]}$. Therefore, for the sake of presentation clarity of the optimization problem, the $v_{k+j|k}$, $j = 0 \dots N-1$, can be considered as the only free variables of $P_N(x_k)$, with the predicted trajectory being implicitly, or, in the linear case, even explicitly, defined by the future inputs.*

The set of initial states $x_0 \in \mathbb{R}^n$, such that $P_N(x_0)$ is feasible, is called the domain of attraction of the controller and represented by $\mathcal{X}_N$. For a $x_k \in \mathcal{X}_N$, the solution of $P_N(x_k)$ and its associated cost are respectively $\hat{\mathbf{v}}^*_{[k,k+N-1]}(x_k)$ and $V_N^*(x_k)$.[4]

[4]The notations $\hat{\mathbf{v}}^*_{[k,k+N-1]}(x_k)$ and $V_N^*(x_k)$ are employed in this work in order to emphasize the dependence of the optimal solution on the current state.

The terminal set $\mathcal{X}_f$ must be an admissible Robust Positively Invariant (RPI) set[5], with an uniformly continuous terminal control law $v_t\colon \mathcal{X}_f \to \mathbb{R}^m$, with $u_t(x) \coloneqq \pi(x, v_t(x))$, $u_t(0) = 0$, such that:

Assumption 2.1 (Terminal Set)**.**

(i) The terminal set is compact and contains the origin as an interior point, where $\mathcal{X}_f \subseteq \mathcal{V}_N = \{x \in \mathbb{R}^n\colon (x, u_t(x)) \in \mathcal{A}_N\}$ *and* $\mathcal{A}_N \subseteq \mathcal{Z}(N) = \{(x, \pi(x, v)) \in \mathbb{R}^{n+m}\colon (x, v) \in \mathcal{Z}_\pi(N)\}$ *is the* N*-step-ahead admissible set.*

(ii) The terminal set is Robust Positively Invariant, such that $f(x, u_t(x)) \oplus \mathcal{S}(N) \subseteq \mathcal{X}_f$ *for all* $x \in \mathcal{X}_f$*.*

Finally, the following typical set of assumptions is imposed to ensure Input-to-State stability:

Assumption 2.2 (Input-to-State Stability)**.**

(i) Let $L_\pi(x, v)$ *be a definite positive, uniformly continuous function such that, for any feasible* x *and* v*:*

$$L_\pi(x, v) \geq \alpha_L(\|x\|), \tag{2.10a}$$

$$|L_\pi(x_1, v_1) - L_\pi(x_2, v_2)| \leq \lambda_x(\|x_1 - x_2\|) + \lambda_v(\|v_1 - v_2\|), \tag{2.10b}$$

where λ_x *and* λ_v *are* $\mathcal{K}$*-functions and* α_L *is a* $\mathcal{K}_\infty$*-function.*

(ii) Let the terminal cost function $V_f(x)$ *be a definite positive, uniformly continuous function in* $\mathcal{X}_f$ *such that, for any* $x \in \mathcal{X}_f$*:*

$$0 \leq V_f(x) \leq \beta_V(\|x\|), \tag{2.11a}$$

$$|V_f(x_1) - V_f(x_2)| \leq \delta(\|x_1 - x_2\|), \tag{2.11b}$$

$$V_f(f_\pi(x, v_t(x))) - V_f(x) \leq -L_\pi(x, v_t(x)), \tag{2.11c}$$

where β_V *is a* $\mathcal{K}_\infty$*-function and* δ *is a* $\mathcal{K}$*-function.*

The receding horizon policy then states that the Model Predictive Control law is given by

$$u_k = \kappa(x_k) = \pi(x_k, v_k^*). \tag{2.12}$$

It is worth noting that, in the context of optimal control theory, the cost term $\sum_{j=0}^{N-1} L_\pi(x_{k+j|k}, v_{k+j|k}) + V_f(x_{k+N|k})$ is referred as a cost functional. This is because

[5] A set $\mathcal{X}_f \subseteq \mathbb{R}^n$ is said Robust Positively Invariant in relation to the system (2.4a), subject to the control law $v_k = v_t(x_k)$ and disturbances $w_k \in \mathcal{W}_f$, if for any $x \in \mathcal{X}_f$ and $w \in \mathcal{W}_f$ we have $f_\pi(x, v_t(x)) + w \in \mathcal{X}_f$.

the future inputs and predicted trajectory, particularly in the continuous-time case, are themselves functions of the time t, being $L_\pi(x, v)$ and $V_f(x)$ *functions of functions* (functionals). In the literature of predictive control of discrete-time systems, however, the nomenclature of cost functions is commonly used, since here the sequence $\hat{\mathbf{v}}_{[k,k+N-1]}$ can be seen as a simple vector, rather than a function of k.

Under these assumptions, Lemma 2.1 and Theorem 2.1 [19, 34] guarantee recursive feasibility and Input-to-State Stability (ISS) of the system (2.1) subject to the control law (2.12).

Lemma 2.1 (Recursive Feasibility [34])**.**
Given $x_k \in \mathcal{X}_N$, then $x_{k+1} = f(x_k, \kappa(x_k)) + w_k \in \mathcal{X}_N$ for all $w_k \in \mathcal{W}$. Furthermore, given $\hat{\mathbf{v}}^*(x_k) = (v^*_{k|k}, \ldots, v^*_{k+N-1|k})$ solution of $P_N(x_k)$, then $\hat{\mathbf{v}}^c = (v^*_{k+1|k}, \ldots, v^*_{k+N-1|k}, v_t(x^*_{k+N|k}))$ defines a feasible (candidate) solution of $P_N(x_{k+1})$.

Proof. Consider the optimal nominal trajectory $x^*_{k+j|k} = \phi_\pi(j, x_k, \hat{\mathbf{v}}^*, \mathbf{0})$ and the candidate solution $\hat{\mathbf{v}}^c = (v^*_{k+1|k}, \ldots, v^*_{k+N-1|k}, v_t(x^*_{k+N|k}))$, which provides the standard one-step ahead candidate predictions $x^c_{k+j|k+1} = \phi_\pi(j, x_{k+1}, \hat{\mathbf{v}}^c, \mathbf{0})$. From Condition 2.1, we have $x^c_{k+j|k+1} \in x^*_{k+j|k} \oplus \mathcal{S}(j-1),\ j = 1 \ldots N$.

Therefore, the constraints $(x^*_{k+j|k}, v^*_{k+j|k}) \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1$ and $x^*_{k+N|k} \in \mathcal{X}_f$ directly imply that $(x^c_{k+1+j|k+1}, v^c_{k+1+j|k+1}) \in (x^*_{k+1+j|k}, v^*_{k+1+j|k}) \oplus \{\mathcal{S}(j) \times 0\} \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1$, due to the admissibility of $\mathcal{X}_f$ ($\mathcal{X}_f \subseteq \mathcal{V}_N$) and the definition of the tighter constraints (2.8), where $v^*_{k+N|k} = v_t(x^*_{k+N|k})$ is defined for simplicity of notation.

For the terminal constraint, notice that the candidate solution is such that $x^c_{k+1+N|k+1} = f_\pi(x^c_{k+N|k+1}, v_t(x^*_{k+N|k}))$. Therefore, since $x^c_{k+N|k+1} - x^*_{k+N|k} \in \mathcal{S}(N-1)$, $x^c_{k+1+N|k+1} \in f_\pi(x^*_{k+N|k}, v_t(x^*_{k+N|k})) \oplus \mathcal{S}(N)$. Finally, since $x^*_{k+N|k} \in \mathcal{X}_f$, then $x^c_{k+1+N|k+1} \in \mathcal{X}_f$ follows from the robust invariance of $\mathcal{X}_f$ and $\hat{\mathbf{v}}^c$ is a feasible candidate solution of $P_N(x_{k+1})$. □

Theorem 2.1 (Input-to-State Stability [34])**.**
The system (2.1) *subject to the NMPC control law* (2.12) *is Input-to-State stable. That is, for any initial state $x_0 \in \mathcal{X}_N$ and disturbances $w_k \in \mathcal{W}$, we have:*

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|\mathbf{w}_{[0,k]}\|), \tag{2.13}$$

where β is a $\mathcal{KL}$-function and γ is a $\mathcal{K}$-function.

Proof. The candidate solution $\hat{\mathbf{v}}^c$ is used to show via standard optimality arguments that $V^*_N(x_k)$ is an ISS-Lyapunov function. From the uniform continuity of the model, there exists a $\mathcal{K}$-function $\sigma_x(\cdot)$ such that $\left\| x^c_{k+j+1|k+1} - x^*_{k+j+1|k} \right\| \leq \sigma^j_x(\|w_k\|),\ j = 0 \ldots N,$

where $v^*_{k+N|k} = v_t(x^*_{k+N|k})$ and $x^*_{k+N+1|k} = f_\pi(x^*_{k+N|k}, v^*_{k+N|k})$ are defined for simplicity of notation. Thus, from Equations (2.10b) and (2.11b), we have

$$|L_\pi(x^c_{k+j|k+1}, v^c_{k+j|k+1}) - L_\pi(x^*_{k+j|k}, v^*_{k+j|k})| \leq \lambda_x(\sigma_x^{j-1}(\|w_k\|)), \quad j = 1 \ldots N,$$
$$|V_f(x^c_{k+N+1|k+1}) - V_f(x^*_{k+N+1|k})| \leq \delta(\sigma_x^N(\|w_k\|)).$$

Therefore, using the fact that the terminal cost is a Lyapunov function of the terminal control law (2.11c), the candidate cost is bounded by

$$\begin{aligned}
V_N^c(x_{k+1}) &= \sum_{j=0}^{N-1} L_\pi(x^c_{k+1+j|k+1}, v^c_{k+1+j|k+1}) + V_f(x^c_{k+1+N|k+1}) \\
&\leq \sum_{j=1}^{N} L_\pi(x^*_{k+j|k}, v^*_{k+j|k}) + V_f(x^*_{k+1+N|k}) + \theta(\|w_k\|) \\
&= \sum_{j=1}^{N-1} L_\pi(x^*_{k+j|k}, v^*_{k+j|k}) + (L_\pi(x^*_{k+N|k}, v^*_{k+N|k}) + V_f(x^*_{k+1+N|k})) + \theta(\|w_k\|) \\
&\leq \sum_{j=1}^{N-1} L_\pi(x^*_{k+j|k}, v^*_{k+j|k}) + V_f(x^*_{k+N|k}) + \theta(\|w_k\|) \\
&= V_N^*(x_k) - L_\pi(x_k, v_k^*) + \theta(\|w_k\|), \\
&\leq V_N^*(x_k) - \alpha_L(\|x_k\|) + \theta(\|w_k\|)
\end{aligned}$$

where $\theta(\|w_k\|) = \sum_{j=0}^{N-1} \lambda_x(\sigma_x^j(\|w_k\|)) + \delta(\sigma_x^N(\|w_k\|))$ is a $\mathcal{K}$-function. The property of decreasing cost is thus ensured, since by optimality $V_N^*(x_{k+1}) \leq V_N^c(x_{k+1})$.

$$V_N^*(x_{k+1}) - V_N^*(x_k) \leq -\alpha_L(\|x_k\|) + \theta(\|w\|). \tag{2.14}$$

Now, notice that $V_N^*(x_k) \geq L_\pi(x_k, v_k^*) \geq \alpha_L(\|x_k\|)$ and, for $x_k \in \mathcal{X}_f$, the unconstrained terminal control law is feasible throughout the entire prediction horizon and by optimality $V_N^*(x_k) \leq V_f(x_k) \leq \beta_V(\|x_k\|)$, $\forall x_k \in \mathcal{X}_f$. Furthermore, from the continuity of the costs and compactness of constraints, $V_N^*(x_k)$ is limited in $\mathcal{X}_N$ and thus, from Lemma B.2, there exists a $\mathcal{K}_\infty$-function $\overline{\beta}_V$ such that:

$$\alpha_L(\|x_k\|) \leq V_N^*(x_k) \leq \overline{\beta}_V(\|x_k\|), \quad \forall x_k \in \mathcal{X}_N. \tag{2.15}$$

Finally, from (2.14) and (2.15), since from recursive feasibility $\mathcal{X}_N$ is a RPI set for the closed-loop system, $V_N^*(x_k)$ is an ISS-Lyapunov function and, through Lemma B.1, the system subject to the NMPC control law is ISS-stable. □

2.5 Simplifying Assumptions

In this section, certain common simplifying assumptions, which facilitate the NMPC design, are presented. First, consider that the state and input constraints are independent and polyhedral, that is, there exist compact sets $\mathcal{X} = \{x \in \mathbb{R}^n : H_x x \leq r_x\}$ and

$\mathcal{U} = \{u \in \mathbb{R}^m \;:\; H_u u \leq r_u\}$, where the matrices H_x and H_u and vectors r_x and r_u define the half-spaces of the polyhedral restrictions, in such a way that $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$, i.e.

$$x_k \in \mathcal{X},\; u_k \in \mathcal{U}, \quad \forall k \in \mathbb{N}. \tag{2.16}$$

This assumption simplifies the constraint tightening process, terminal set computation and optimization algorithm and encompass most practical applications. For instance, any restrictions of the type $x_j^{min} \leq x_j \leq x_j^{max}$ and $u_i^{min} \leq u_i \leq u_i^{max}$ can be written as polyhedral constraints (2.16). For the feedback law $\pi\colon \mathbb{R}^{n+m} \to \mathbb{R}^m$, in order to maintain the linearity of the constraints when the virtual inputs are considered, it is interesting to use a linear feedback, that is

$$u_k = \pi(x_k, v_k) = v_k + K_v x_k, \quad \forall k \in \mathbb{N}, \tag{2.17}$$

where the matrix $K_v \in \mathbb{R}^{m\times n}$ defines the relationship between virtual and real input[6]. Hence, the restrictions on state and virtual input are still polyhedral, and a direct description of $\mathcal{Z}_\pi$ is given by

$$\mathcal{Z}_\pi = \left\{ z \in \mathbb{R}^{n+m}\colon \begin{pmatrix} H_x & 0 \\ H_u K_v & H_u \end{pmatrix} z \leq \begin{pmatrix} r_x \\ r_u \end{pmatrix} \right\}. \tag{2.18}$$

For the controller design, quadratic stage and terminal costs are often chosen, with $L_\pi(x, v) = x^\intercal Q x + u^\intercal R u$, $u = v + K_v x$, and $V_f(x) = x^\intercal P x$. Notice that, in this case, the positive definiteness and uniform continuity of the cost functions are guaranteed for any $Q, P \succ 0 \in \mathbb{R}^{n\times n}$ and $R \succeq 0 \in \mathbb{R}^{m\times m}$. Furthermore, $L_\pi(x, v) \geq \lambda_{Q,m} \|x\|_2^2$ and $V_f(x) \leq \lambda_{P.M} \|x\|_2^2$, where $\lambda_{Q,m} > 0$ is the smallest eigenvalue of Q and $\lambda_{P,M} > 0$ is the biggest eigenvalue of P, and thus Eqs. (2.10a) and (2.11a) are also satisfied.

Finally, considering a linear terminal control law $u_t(x) = K_t x$, the decreasing terminal cost assumption (2.11c) can be rewritten in terms of matrix inequalities, as shown in Theorem 2.2, adapted for the discrete-time case from [4, Section 5.1].

Theorem 2.2. *Consider the nonlinear system* (2.1)*, the terminal control law* $u_t(x) = K_t x$ *and the stage and terminal costs* $L_\pi(x, v) = x^\intercal Q x + u^\intercal R u$, $V_t(x) = x^\intercal P x$, *and let* $\mathbf{A} \in \mathbb{I}^{n\times n}$, $\mathbf{B} \in \mathbb{I}^{n\times m}$ *be interval matrices satisfying* $\nabla_x^\intercal f(\mathcal{A}_N) \in \mathbf{A}$, $\nabla_u^\intercal f(\mathcal{A}_N) \in \mathbf{B}$.

If, for any $A_\ell \in \mathbf{A}$ *and* $B_\ell \in \mathbf{B}$, *we have:*

$$(A_\ell + B_\ell K_t)^\intercal P (A_\ell + B_\ell K_t) - P + (Q + K_t^\intercal R K_t) \preceq 0, \tag{2.19}$$

then the decreasing cost assumption (2.11c) *is satisfied for all* $x \in \mathcal{V}_N$.

[6] A method for choosing K_v so as to mitigate the disturbance propagation is presented in Appendix A.

Proof. Given $x_k \in \mathcal{V}_N$, with $u_t(x_k) = K_t x_k$ and $x_{k+1} = f(x_k, u_t(x_k))$, then the inequation (2.11c) can be rewritten as

$$x_{k+1}^\intercal P x_{k+1} - x_k^\intercal P x_k \leq -x_k^\intercal (Q + K_t^\intercal R K_t) x_k.$$

Additionally, through the mean-value theorem and the definition of $\mathcal{V}_N$, there exist $A_\ell \in \mathbf{A}$, $B_\ell \in \mathbf{B}$ such that:

$$\begin{aligned} f(x_k, u_t(x_k)) &= f(0,0) + A_\ell x_k + B_\ell u_t(x_k) \\ &= (A_\ell + B_\ell K_t) x_k. \end{aligned}$$

Therefore, $x_{k+1} = (A_\ell + B_\ell K_t) x_k$, and Eq. (2.11c) is equivalent to:

$$\begin{aligned} x_{k+1}^\intercal P x_{k+1} - x_k^\intercal P x_k &\leq -x_k^\intercal (Q + K_t^\intercal R K_t) x_k, \\ x_k^\intercal (A_\ell + B_\ell K_t)^\intercal P (A_\ell + B_\ell K_t) x_k - x_k^\intercal P x_k &\leq -x_k^\intercal (Q + K_t^\intercal R K_t) x_k, \\ x_k^\intercal ((A_\ell + B_\ell K_t)^\intercal P (A_\ell + B_\ell K_t) - P) x_k &\leq -x_k^\intercal (Q + K_t^\intercal R K_t) x_k, \end{aligned}$$

which, based on the matrix inequality (2.19), is satisfied. □

A par of matrices (K_t, P) satisfying (2.19) can be computed from the vertices of $\mathbf{A}$ and $\mathbf{B}$ via Linear Time-Varying (LTV) control strategies [15], considering the LTV system $x_{k+1} = A_\ell x_k + B_\ell u_k$, where A_ℓ and B_ℓ are a convex combination of the vertices of $\mathbf{A}$ and $\mathbf{B}$, respectively.

2.6 Terminal Sets

In order to implement a robust predictive controller as proposed in Section 2.4, a terminal robust positively invariant (RPI) set $\mathcal{X}_f \subseteq \mathcal{V}_N$ is necessary. In this section, an algorithm for calculating polyhedral RPI sets [14] based on the definition of precursor sets is presented.

Definition 2.1 (Precursor Set). *Given an autonomous system $x_{k+1} = f_t(x_k)$ and a set $\mathcal{X}_t \subseteq \mathbb{R}^n$, the precursor of $\mathcal{X}_t$ is defined by the set of states that are steered by the system to $\mathcal{X}_t$, i.e.*

$$Pre(\mathcal{X}_t) = \{x \in \mathbb{R}^n \colon f_t(x) \in \mathcal{X}_t\}. \tag{2.20}$$

Consider a linear system $x_{k+1} = A_t x_k + w_k$, with $w_k \in \mathcal{W}_t$, and a polyhedral set $\mathcal{O}_0 = \{x \in \mathbb{R}^n \colon H^0 x \leq r^0\} \subseteq \mathcal{V}_N$. A sequence of sets $\mathcal{O}_k$ can then be created by induction from $\mathcal{O}_0$ by $\mathcal{O}_{k+1} = Pre(\mathcal{O}_k \ominus \mathcal{W}_t) \cap \mathcal{O}_k$. Notice that, given a vector γ^k satisfying $\gamma_j^k = \max_{w \in \mathcal{W}_t} H^0_{j,:} A_t^k w$, we have[7]

$$\mathcal{O}_{k+1} = \left\{ x \in \mathbb{R}^n \colon \begin{pmatrix} H^k \\ H^0 A_t^k \end{pmatrix} x \leq \begin{pmatrix} r^k \\ r^k - \gamma^k \end{pmatrix} \right\} \tag{2.21}$$

[7]Notice that, if $\mathcal{W}_t$ is a zonotope, γ_k can be obtained algebraically as described in Appendix A.

and the sequence $\mathcal{O}_k$ can be calculated with low computational cost[8]. Notice that, since $\mathcal{O}_{k+1} \subseteq \mathcal{O}_k$, $\forall k \in \mathbb{N}$, each set $\mathcal{O}_k$ is admissible ($\mathcal{O}_k \subseteq \mathcal{V}_N$). Furthermore, from the definition of precursor sets, if for some $N \in \mathbb{N}$ we have $\mathcal{O}_{N+1} = \mathcal{O}_N$, i.e. $\mathcal{O}_N \subseteq Pre(\mathcal{O}_N \ominus \mathcal{W}_t)$, then $\mathcal{O}_N$ is a robust positively invariant set for the system $x_{k+1} = A_t x_k + w_k$ with disturbances $w_k \in \mathcal{W}_t$. For the linear case, this recursive method can thus be used to obtain an admissible RPI set.

Remark 2.3. *If the linear system defined by the matrix A_t is stable, a finite $N \in \mathbb{N}$ such that $\mathcal{O}_{N+1} = \mathcal{O}_N$ exists. However, the $\mathcal{O}_N$ obtained might be empty. A more detailed discussion of this algorithm, can be found in [14].*

For the general case of nonlinear systems, a linearized model can be considered, with the nonlinearities treated as additional disturbances. Given the nonlinear system $x_{k+1} = f_t(x_k) + w_k$, with $w_k \in \mathcal{W}_t$, a linear model $x_{k+1} = A_t x_k + \tilde{w}_k$ is considered, with extended disturbances $\tilde{w}_k \in \mathcal{W}_{ext} = \mathcal{W}_t \oplus \mathcal{W}_{nl}$, where $\mathcal{W}_{nl}$ bounds the deviation between the nonlinear and linearized models, i.e.

$$\delta(x) = f_t(x) - A_t x \in \mathcal{W}_{nl}, \quad \forall x \in \mathcal{V}_N. \tag{2.22}$$

However, due to the conservatism brought by considering the nonlinearities as additive disturbances, the application of this method to obtain RPI sets for nonlinear systems may result in conservative, or even empty, sets.

In order to reduce this source of conservatism, the set $\mathcal{V}_N$ can be scaled by a parameter $\lambda \in (0, 1]$, resulting in a family of sets $\mathcal{V}_N(\lambda) = \lambda \mathcal{V}_N$. From this contraction of the terminal constraint, the deviation between nonlinear model and linearized system is reduced. In particular, a $\mathcal{K}$-function $\alpha(\lambda)$ can be obtained in such a way that

$$\mathcal{W}_{nl}(\lambda) \subseteq \alpha(\lambda)\mathcal{W}_{nl}. \tag{2.23}$$

In general, α is *little-o* of λ [9] and the sets $\mathcal{W}_{nl}(\lambda)$ decrease faster than the $\mathcal{V}_N(\lambda)$. Outer limits for the minimal Robust Positively Invariant (mRPI) sets of the linearized system subjected to the disturbances $\mathcal{W}_t$ and $\mathcal{W}_{nl}$ (respectively $\mathcal{R}^t_\infty$ and $\mathcal{R}^{nl}_\infty$) are then calculated. The existence of a RPI set for the linearized system subject to constraints $\mathcal{V}_N(\lambda)$ and disturbance $\mathcal{W}_{amp}(\lambda) = \mathcal{W}_t \oplus \mathcal{W}_{nl}(\lambda)$ is thus equivalent to the condition

$$\mathcal{R}^t_\infty \oplus \alpha(\lambda)\mathcal{R}^{nl}_\infty \subseteq \lambda \mathcal{V}_N. \tag{2.24}$$

Therefore, the maximal value of $\lambda \in (0, 1]$, such that (2.24) is satisfied (if it exists), is searched, and a non-empty terminal set $\mathcal{X}_f$ can then be obtained from $\mathcal{V}_N(\lambda^*)$ and

[8]Constraint reduction methods can be used in order to limit the growth of the number of halfspaces of the polyhedral sets.

[9]A function $\alpha(\lambda)$ is said to be *little-o* of λ if $\lim_{\lambda\to 0} \frac{\alpha(\lambda)}{\lambda} = 0$.

$\mathcal{W}_{amp}(\lambda^*)$. This scaling approach is a contribution of this project and was first proposed in the related paper [6]. It generalizes the method of calculating polyhedral RPI sets for nonlinear systems based on linear approximations presented in [14].

2.7 Zonotopes

Zonotopes are a particular class of convex and symmetric polytopes [10]. They can be represented as the Minkowsky sum of line segments or, alternatively, as the affine image of an unitary box $\mathcal{B}_\infty^{n_g}$, as follows

$$Z = \{G, c\} = c \oplus G\mathcal{B}_\infty^{n_g}, \tag{2.25}$$

where $c \in \mathbb{R}^n$ is the center, and the columns of $G \in \mathbb{R}^{n \times n_g}$, which are assumed without loss of generality to be linearly independent, are the generators of the zonotope. The number of generators $n_g \geq n$ is associated to the complexity of the zonotope, with $n_g = n$ in parallelotopes. A zonotope is said centered when its center is the origin ($c = 0$).

The application of zonotopes on state estimation is partially due to the simplicity and efficiency of linear transformations and Minkowski sums of zonotopes [1, 35]. Given $Z_1 = \{G_1, c_1\},\ Z_2 = \{G_2, c_2\} \subseteq \mathbb{R}^n$, $R \in \mathbb{R}^{m \times n}$, we have

$$RZ_1 = \{RG_1, Rc_1\}, \tag{2.26a}$$

$$Z_1 \oplus Z_2 = \left\{ \begin{pmatrix} G_1 & G_2 \end{pmatrix}, c_1 + c_2 \right\}. \tag{2.26b}$$

Therefore, such operations can be made algebraically, with low computational cost. Furthermore, efficient methods for calculating the Pontryagin difference of a polytope by a zonotope [2] and simplifying zonotopes (reducing the number of generators) [10, 35] are presented in Appendix A. Zonotopes are applied in this project for the computation of less conservative disturbance propagation sets $\mathcal{S}(j)$ and consequent constraint tightening, as detailed in Chapter 4.

Recapitulation

In this chapter, a review of the *state-of-the-art* on robust nonlinear model predictive control based on nominal predictions [34, 19] was presented. Tightened constraints, derived from disturbance propagation sets, were applied to the nominal predictions in order to ensure robust constraint satisfaction. In particular, the following topics were discussed:

- System description and closed-loop prediction: The general state-space model of a nonlinear dynamic system with additive disturbances was presented, and a virtual input was introduced in a closed-loop prediction paradigm, with the goal of reducing the disturbance propagation through the system dynamics.

- Disturbance propagation: Sets $\mathcal{S}(j)$ which limit the disturbance propagation were defined, such that $x_{k+j|k+1} - x_{k+j|k} \in \mathcal{S}(j-1),\ \forall j = 1 \dots N+1$. Based on these sets, the tightened constraints were recursively calculated.

- NMPC optimization problem and control law: The NMPC optimization problem was presented, with the control law being defined from the receding horizon policy. Under typical assumptions on the cost functions and terminal control law and set, recursive feasibility and input-to-state stability guarantees were presented [34, 19].

- Simplifying assumptions: Certain common assumptions on the constraints, costs and control law were presented, and it was shown how they can simplify the controller design.

- Terminal sets: An algorithm to compute polyhedral RPI sets of nonlinear systems based on linear approximations [14] was presented. A scaling approach, which reduces the conservatism present in the linearization, was also introduced [6].

- Zonotopes: Zonotopes and their related operations [10] were briefly presented as interesting tools for state estimation and, as will be shown in Chapter 4, disturbance propagation.

Chapter 3

Constant Disturbance Attenuation

The controller presented in Chapter 2 is recursively feasible and Input-to-State stable in the presence of disturbances $w_k \in \mathcal{W}$. However, it presents a typical steady-state offset if the disturbance mean-value is non-zero, due to two undesired effects: (i) Prediction error; and (ii) Objective function bias due to the steady-state target mismatch.

In this chapter, a modified NMPC, which avoids these regulation problems, will be developed, based on the incorporation of a constant disturbance model on the prediction and target correction, establishing a reachable equilibrium in the presence of the constant disturbance.

3.1 Equilibrium with Constant Disturbances

Consider that the additive disturbance is given by $w_k = \mu + \overline{w}_k$, where μ represents the constant portion of the disturbance and $\lim_{k\to\infty} \overline{w}_k = 0$. The output steady-state condition in the presence of constant disturbances may be described by

$$\begin{aligned} \overline{x}_y^\mu &= f_\pi(\overline{x}_y^\mu, \overline{v}_y^\mu) + \mu, \\ \overline{y} &= h_\pi(\overline{x}_y^\mu, \overline{v}_y^\mu). \end{aligned} \tag{3.1}$$

Due to the disturbance effect, a modified steady-state target satisfying

$$\begin{aligned} \overline{x}_0^\mu &= f_\pi(\overline{x}_0^\mu, \overline{v}_0^\mu) + \mu, \\ 0 &= h_\pi(\overline{x}_0^\mu, \overline{v}_0^\mu) \end{aligned} \tag{3.2}$$

is sought, in order to regulate the output to the origin. Notice that if $\mu = 0$, then $\overline{x}_0^\mu = 0$, $\overline{v}_0^\mu = 0$, and the nominal prediction case is recovered.

Assumption 3.1. *For a given constant disturbance $\mu \in \mathcal{W}$, consider that there exists a unique corrected steady-state $\overline{x} = g_x(\mu)$, $\overline{v} = g_v(\mu)$, where $g_x\colon \mathcal{W} \to \mathbb{R}^n$ and $g_v\colon \mathcal{W} \to \mathbb{R}^m$*

are Lipschitz continuous, such that:

$$
\begin{aligned}
\overline{x} &= f_\pi(\overline{x}, \overline{v}) + \mu, \\
0 &= h_\pi(\overline{x}, \overline{v}).
\end{aligned} \tag{3.3}
$$

Remark 3.1. *From the implicit function theorem [18], Assumption 3.1 is satisfied if $m = p$ and the following matrix is nonsingular*

$$
\begin{pmatrix} A_\pi(\overline{x}, \overline{v}) - I_n & B_\pi(\overline{x}, \overline{v}) \\ C_\pi(\overline{x}, \overline{v}) & D_\pi(\overline{x}, \overline{v}) \end{pmatrix},
$$

where A_π, B_π, C_π and D_π represent the linearized system (2.4a) *at $(\overline{x}, \overline{v})$, for all $(\overline{x}, \overline{v})$ which satisfy $\overline{x} = f_\pi(\overline{x}, \overline{v}) + \mu$ and $h_\pi(\overline{x}, \overline{v}) = 0$, for some $\mu \in \mathcal{W}$.*

In the case of $m > p$, additional constraints on steady state or input can be imposed to make the correspondence $\mu \to (\overline{x}, \overline{v})$ unique. For the case $m < p$, due to the lack of degrees of freedom, all outputs cannot be simultaneously regulated to the origin and a modified output function, namely $\tilde{y}_k = h_m(x_k, u_k)$, with $h_m \colon \mathbb{R}^{n \times m} \to \mathbb{R}^m$, should be specified to replace the original one in order to characterize an alternative reachable output target.

3.2 Constant Disturbance Estimation

In order to incorporate the constant disturbance model on the prediction and calculate the modified target (3.1), an estimation of the disturbance mean-value μ_k is necessary. The effective disturbance value at $k-1$ can be directly obtained at time k as $w_{k-1} = x_k - f_\pi(x_{k-1}, v_{k-1})$. A Bounded-Input-Bounded-Output (BIBO) stable low-pass filter with unitary static gain can then be used to estimate the disturbance mean-value, thus attenuating undesired high-frequency noise. The estimated disturbance, namely $\hat{\mu}_k$, is then a filtered version of w_k. As an example, it can be obtained through a simple first-order low-pass filter, i.e.

$$
\hat{\mu}_k = a\hat{\mu}_{k-1} + (1-a)w_{k-1}, \tag{3.4}
$$

where $0 < a < 1$ is a free design parameter. Hence, a modified steady-state target at k can be obtained from $\hat{\mu}_k$ as $\hat{x}^\mu_{0,k} = g_x(\hat{\mu}_k)$, $\hat{v}^\mu_{0,k} = g_v(\hat{\mu}_k)$, defining a new regulatory objective such that $\lim_{k\to\infty} y_k = 0$.

Notice that, from the uniform continuity of $g_x(\cdot)$ and $g_v(\cdot)$, the modified targets are bounded by the estimated disturbances, such that

$$
\begin{aligned}
\left\|\Delta\hat{x}^\mu_{0,k}\right\| &\le \rho_{dx}(\|\Delta\hat{\mu}_k\|), \quad \left\|\hat{x}^\mu_{0,k}\right\| \le \rho_x(\|\hat{\mu}_k\|), \\
\left\|\Delta\hat{v}^\mu_{0,k}\right\| &\le \rho_{dv}(\|\Delta\hat{\mu}_k\|), \quad \left\|\hat{v}^\mu_{0,k}\right\| \le \rho_v(\|\hat{\mu}_k\|),
\end{aligned} \tag{3.5}
$$

where $\rho_{dx}(\cdot)$, $\rho_x(\cdot)$, $\rho_{dv}(\cdot)$ and $\rho_v(\cdot)$ are $\mathcal{K}$-functions.

Considering the mean-value estimation, the following disturbance sets, besides the direct bounds on w_k ($w_k \in \mathcal{W}$), are considered: (i) $\hat{\mu}_k \in \mathcal{M}$, (ii) $\hat{\mu}_{k+1} - \hat{\mu}_k \in \mathcal{DM}$, and (iii) $w_k - \hat{\mu}_k \in \overline{\mathcal{W}}$. For simplicity, $\tilde{w}_k = w_k - \hat{\mu}_k$ is defined. Notice that, if $\hat{\mu}_k = 0,\ \forall\, k \geq 0$ is enforced, then $\mathcal{M} = \mathcal{DM} = \{0\}$ and $\overline{\mathcal{W}} = \mathcal{W}$, such that the case without offset compensation presented in Chapter 2 is recovered. Indeed, this result is an extension to handle the undesired constant disturbance effects.

Being $\mathbf{F}(z)$ the filter transfer matrix from w_k to $\hat{\mu}_k$, the auxiliary sets $\mathcal{M}$, $\mathcal{DM}$ and $\overline{\mathcal{W}}$ can be directly obtained from $\mathcal{W}$ and $\mathbf{F}(z)$. First, notice that the transfer functions from w_k to $\hat{\mu}_{k+1} - \hat{\mu}_k$ and from w_k to $w_k - \hat{\mu}_k$ are respectively $(z-1)\mathbf{F}(z)$ and $\mathbf{I} - \mathbf{F}(z)$. Therefore, we have

$$\mathcal{M} = |\mathbf{F}(z)|_1 \mathcal{W}, \quad \mathcal{DM} = |(z-1)\mathbf{F}(z)|_1 \mathcal{W}, \quad \overline{\mathcal{W}} = |\mathbf{I} - \mathbf{F}(z)|_1 \mathcal{W}, \tag{3.6}$$

where $|\mathbf{H}(z)|_1$ stands for the absolute sum of the impulse response of $\mathbf{H}(z)$. In the particular case of a first-order filter, $\mathbf{F}(z) = \frac{(1-a)}{z-a}\mathbf{I}$ and $\mathcal{M} = \mathcal{W}$, $\mathcal{DM} = 2(1-a)\mathcal{W}$, $\overline{\mathcal{W}} = 2\mathcal{W}$. Notice that, as expected, the size of $\mathcal{DM}$ is highly dependent on the value of a, with a smaller $\mathcal{DM}$ related to a slower variation of $\hat{\mu}_k$ ($a \to 1$). Smaller sets $\mathcal{DM}$ are desirable for the disturbance propagation, as discussed in Chapter 4, but this comes at the cost of slower model update and convergence.

Remark 3.2. *It is worth noting that the mean-value disturbance estimate used in the prediction is not necessarily a filtered version of the disturbance, since, for* offset *correction purposes, it is only required that it converges in steady-state to the disturbance mean. As an example, the difference $\hat{\mu}_{k+1} - \hat{\mu}_k$ can be artificially limited in order to reduce the set $\mathcal{DM}$, via minimizing the difference $\|\hat{\mu}_k - \overline{\mu}_k\|$ subject to $\hat{\mu}_k \in \mathcal{M}$ and $\hat{\mu}_k \in \mathcal{DM}$, where $\overline{\mu}_k$ is the output of the filter.*

3.3 Disturbance Propagation

As in the case of the controller described in Chapter 2, predictions of the system trajectory are necessary for the formulation of the NMPC optimization problem. Here, however, instead of nominal predictions, a constant disturbance model is incorporated, resulting in the predictions

$$x_{k+j|k} = \phi_\pi(j, x_k, \hat{\mathbf{v}}_{[k,k+j-1]}, \hat{\mathbf{w}}_k), \quad j \geq 0, \tag{3.7}$$

where the future disturbances are considered to be constant and equal to the mean-value estimate at k[1]. The disturbance propagation sets $\mathcal{S}(j)$ must then be modified in order to

[1]Notice that in the case of $\hat{\mu}_k = 0$, $\forall k \in \mathbb{N}$, i.e. no steady-state disturbance, nominal predictions are recovered from Eq. (3.7).

limit the difference between the predictions made at k and $k+1$, taking into consideration the constant disturbance model and its actualization. Therefore, they must satisfy the following condition:

Condition 3.1.

(i) $\mathcal{S}(0)$ *is a compact set that contains* $\overline{\mathcal{W}}$.

(ii) $\mathcal{S}(j),\ j = 1 \dots N$, *is a compact set such that for all* $x_a,\ x_b \in \mathbb{R}^n,\ \mu_a,\ \mu_b \in \mathcal{M}$ *and* $v \in \mathbb{R}^m$ *with* $(x_a, v) \in \mathcal{Z}_\pi \ominus (\mathcal{S}(j-1) \times \{0\})$, $\mu_b - \mu_a \in \mathcal{DM}$ *and* $x_b - x_a \in \mathcal{S}(j-1)$, *we have* $(f_\pi(x_b, v) + \mu_b) - (f_\pi(x_a, v) + \mu_a) \in \mathcal{S}(j)$.

Considering $x_a = x_{k+1|k} = f_\pi(x_k, v_k) + \hat{\mu}_k$ and $x_b = x_{k+1}$, we then have $x_b - x_a \in \overline{\mathcal{W}} \subseteq \mathcal{S}(0)$ and thus, from induction, $x_{k+j|k+1} \in x_{k+j|k} \oplus \mathcal{S}(j-1),\ j = 1 \dots N+1$, for any admissible control sequence $\hat{\mathbf{v}}_{[k,k+N]}$ and estimated disturbance means $\hat{\mu}_k,\ \hat{\mu}_{k+1}$. Notice that Condition 2.1 is a particular case of Condition 3.1 when there is no model correction ($\hat{\mu}_k = 0,\ \forall k \in \mathbb{N}$).

3.4 NMPC with Constant Disturbance Attenuation

This section develops a NMPC control algorithm based on nonlinear predictions with the incorporation of a constant disturbance model (3.7). The control design is similar to the one presented in Section 2.4, but adapted in a way to incorporate the mean-value estimates $\hat{\mu}_k$ and the steady-state target correction.

Therefore, at each sampling instant, the state x_k is obtained, the disturbance mean-value $\hat{\mu}_k$ is estimated, and the following optimization problem $P_N^\mu(x_k, \hat{\mu}_k)$ is solved:

$$\min_{\hat{\mathbf{v}}_{[k,k+N-1]}} \sum_{j=0}^{N-1} L_\pi(x_{k+j|k} - \hat{x}^\mu_{0,k}, v_{k+j|k} - \hat{v}^\mu_{0,k}) + V_f(x_{k+N|k} - \hat{x}^\mu_{0,k}) \tag{3.8a}$$

$s.t:$

$$x_{k+j+1|k} = f_\pi(x_{k+j|k}, v_{k+j|k}) + \hat{\mu}_k, \qquad j \in \mathbb{Z}_{[0,N-1]}, \tag{3.8b}$$

$$(x_{k+j|k}, v_{k+j|k}) \in \mathcal{Z}_\pi(j), \qquad j \in \mathbb{Z}_{[0,N-1]}, \tag{3.8c}$$

$$x_{k+N|k} \in \mathcal{X}_f, \tag{3.8d}$$

where $\hat{\mathbf{v}}_{[k,k+N-1]}$ once again represents the future virtual inputs to be chosen, $(\hat{x}^\mu_{0,k}, \hat{v}^\mu_{0,k})$ is the corrected target, and $L_\pi(\cdot,\cdot)$ and $V_f(\cdot)$ are respectively the stage and terminal costs. The tightened constraints $\mathcal{Z}_\pi(j)$ are recursively computed by (2.8), with the disturbance propagation sets $\mathcal{S}(j)$ satisfying Condition 3.1.

For a given estimated disturbance $\hat{\mu}_0$, the set of initial states $x_0 \in \mathbb{R}^n$ which provide a feasible solution to problem (3.8) is the domain of attraction, represented by $\mathcal{X}_N(\hat{\mu}_0)$. For $x_k \in \mathcal{X}_N(\hat{\mu}_k)$, the solution of $P_N^{\mu}(x_k, \hat{\mu}_k)$ and its associated cost are respectively given by $\hat{\mathbf{v}}^*_{[k,k+N-1]}(x_k, \hat{\mu}_k)$ and $V_N^*(x_k, \hat{\mu}_k)$. For the sake of simplicity, the first mean-value estimate is considered to be zero, i.e. $\hat{\mu}_0 = 0$.

As in the case without constant disturbance attenuation, the terminal set $\mathcal{X}_f$ must be an admissible Robust Positively Invariant (RPI) set, with an uniformly continuous terminal control law $v_t \colon \overline{\mathcal{X}_f} \to \mathbb{R}^{n_u}$, with $u_t(x) \coloneqq \pi(x, v_t(x))$, $u_t(0) = 0$, such that:[2]

Assumption 3.2 (Robust Invariant Set)**.**

(i) The terminal set is compact and contains the origin as an interior point, where $\mathcal{X}_f \subseteq \mathcal{V}_N = \{x \in \mathbb{R}^n \colon (x, u_t(x)) \in \mathcal{A}_N\}$ *and* $\mathcal{A}_N \subseteq \mathcal{Z}(N) = \{(x, \pi(x, v)) \in \mathbb{R}^{n+m} \colon (x, v) \in \mathcal{Z}_\pi(N)\}$ *is the* N*-step-ahead admissible set.*

(ii) The terminal set $\mathcal{X}_f$ *satisfies:*

$$f(x, u_t(x)) \oplus \mathcal{M} \oplus \mathcal{S}(N) \subseteq \mathcal{X}_f, \ \forall x \in \mathcal{X}_f. \tag{3.9}$$

Finally, the following typical set of assumptions are imposed to ensure Input-to-State stability:

Assumption 3.3 (Input-to-State Stability)**.**

(i) Let $L_\pi(x, v)$ *be a definite positive, uniformly continuous function such that, for any feasible* x *and* v*:*

$$L_\pi(x, v) \geq \alpha_L(\|x\|), \tag{3.10a}$$

$$|L_\pi(x_1, v_1) - L_\pi(x_2, v_2)| \leq \lambda_x(\|x_1 - x_2\|) + \lambda_v(\|v_1 - v_2\|), \tag{3.10b}$$

where λ_x *and* λ_v *are* $\mathcal{K}$*-functions and* α_L *is a* $\mathcal{K}_\infty$*-function.*

(ii) Let the terminal cost function $V_f(x)$ *be a definite positive, uniformly continuous function in* $\overline{\mathcal{X}_f}$ *such that, for any* $x \in \overline{\mathcal{X}_f}$*:*

$$0 \leq V_f(x) \leq \beta_V(\|x\|), \tag{3.11a}$$

$$|V_f(x_1) - V_f(x_2)| \leq \delta(\|x_1 - x_2\|), \tag{3.11b}$$

$$V_f(f_\pi(x, v_t(x))) - V_f(x) \leq -L_\pi(x, v_t(x)), \tag{3.11c}$$

where β_V *is a* $\mathcal{K}_\infty$*-function and* δ *is a* $\mathcal{K}$*-function.*

[2]The set $\overline{\mathcal{X}_f}$ is defined as $\overline{\mathcal{X}_f} = \{x \in \mathbb{R}^n \colon x = x_f - g_x(\mu), \ x_f \in \mathcal{X}_f, \mu \in \mathcal{M}\} \supseteq \mathcal{X}_f$.

Notice that these assumptions are analogous to the ones presented in Section 2.4, with the small distinctions that $\mathcal{X}_f$ must be a RPI set for disturbances $\overline{w}_k \in \mathcal{M} \oplus \mathcal{S}(N)$ and (3.11c) must be satisfied for all $x \in \overline{\mathcal{X}_f}$.

From the receding horizon policy, the NMPC control law is thus given by

$$u_k = \kappa_\mu(x_k, \hat{\mu}_k) = \pi(x_k, v_k^*), \tag{3.12}$$

where v_k^* is obtained from the solution of $P_N^\mu(x_k, \hat{\mu}_k)$. The proposed strategy is then formally able to deal with the main undesired effects of the steady-state constant disturbance, as prediction error and equilibrium target are corrected in steady-state.

Recursive feasibility and Input-to-State stability are then assured through Lemma 3.1 and Theorem 3.1.

Lemma 3.1 (Recursive Feasibility).
Given $x_k \in \mathcal{X}_N(\hat{\mu}_k)$, then $x_{k+1} = f(x_k, \kappa_\mu(x_k, \hat{\mu}_k)) + w_k \in \mathcal{X}_N(\hat{\mu}_{k+1})$ for all $w_k \in \mathcal{W}$, $\hat{\mu}_k, \hat{\mu}_{k+1} \in \mathcal{M}$, $\hat{\mu}_{k+1} - \hat{\mu}_k \in \mathcal{DM}$, $w_k - \hat{\mu}_k \in \overline{\mathcal{W}}$.

Furthermore, given the optimal sequence virtual inputs $\hat{\mathbf{v}}^*(x_k, \hat{\mu}_k) = (v_0^*, \dots, v_{N-1}^*)$, then $\hat{\mathbf{v}}^c = (v_1^*, \dots, v_{N-1}^*, v_t(x_{k+N|k}^*))$ defines a feasible (candidate) solution of $P_N^\mu(x_{k+1}, \hat{\mu}_{k+1})$.

Proof. Consider the optimal solution at k, $\hat{\mathbf{v}}^*_{[k,k+N-1]}$, and the candidate solution at $k+1$, $\hat{\mathbf{v}}^c = (v_{k+1|k}^*, \dots, v_{k+N-1|k}^*, v_t(x_{k+N|k}^*))$, which provide the optimal and candidate predictions given by $x_{k+j|k}^* = f_\pi(x_{k+j-1|k}^*, v_{k+j-1|k}^*) + \hat{\mu}_k$ and $x_{k+j|k+1}^c = f_\pi(x_{k+j-1|k+1}^c, v_{k+j-1|k+1}^c) + \hat{\mu}_{k+1}$, respectively. Given $x_{k+1|k+1}^c = x_{k+1} = f_\pi(x_k, v_k^*) + w_k$ combined with (i) $\tilde{w}_k = w_k - \hat{\mu}_k$, and (ii) $x_{k+1|k}^* = f_\pi(x_k, v_k^*) + \hat{\mu}_k$, then $x_{k+1|k+1}^c - x_{k+1|k}^* = \tilde{w}_k \in \overline{\mathcal{W}}$. Now, based on Condition 2.1, $x_{k+j|k+1}^c \in x_{k+j|k}^* \oplus \mathcal{S}(j-1),\ j = 1 \dots N$ is ensured.

Therefore, the constraints $(x_{k+j|k}^*, v_{k+j|k}^*) \in \mathcal{Z}_\pi(j),\ j = 0 \dots N-1$ and $x_{k+N|k}^* \in \mathcal{X}_f$ directly imply that $(x_{k+1+j|k+1}^c, v_{k+1+j|k+1}^c) \in (x_{k+1+j|k}^*, v_{k+1+j|k}^*) \oplus \{\mathcal{S}(j) \times 0\} \in \mathcal{Z}_\pi(j),\ j = 0 \dots N-1$ due to the condition $\mathcal{X}_f \subseteq \mathcal{V}_N$ and the definition of the tighter constraints (2.8), where $v_{k+N|k}^* = v_t(x_{k+N|k}^*)$ is defined for simplicity of notation.

For the terminal constraint, we use the fact that $\mathcal{X}_f$ is defined as a robust admissible invariant set (3.9). Notice that the candidate solution is such that $x_{k+1+N|k+1}^c = f_\pi(x_{k+N|k+1}^c, v_t(x_{k+N|k}^*)) + \hat{\mu}_{k+1}$. Therefore, $x_{k+1+N|k+1}^c \in f_\pi(x_{k+N|k}^*, v_t(x_{k+N|k}^*)) + \hat{\mu}_{k+1} \oplus \mathcal{S}(N)$. Hence, as $x_{k+N|k}^* \in \mathcal{X}_f$, then $x_{k+1+N|k+1}^c \in \mathcal{X}_f$ because $f_\pi(x_{k+N|k}^*, v_t(x_{k+N|k}^*)) \oplus \mathcal{M} \oplus \mathcal{S}(N) \subseteq \mathcal{X}_f$ from the terminal set definition. In summary, $\hat{\mathbf{v}}^c$ is a feasible candidate, which completes the recursive feasibility proof. □

Theorem 3.1 (Input-to-State Stability).
System (2.1) subject to the MPC control law (3.12) is input-to-state stable. That is, for any initial state $x_0 \in \mathcal{X}_N(0)$ subject to $w_k \in \mathcal{W}$, $\hat{\mu}_k \in \mathcal{M}$, $\hat{\mu}_{k+1} - \hat{\mu}_k \in \mathcal{DM}$, $w_k - \hat{\mu}_k \in \overline{\mathcal{W}}$, $\forall k \geq 0$, then

$$\left\| x_k - \hat{x}_{0,k}^\mu \right\| \leq \beta(\left\| x_0 - \hat{x}_{0,0}^\mu \right\|, k) + \gamma(\left\| \mathbf{w}_{[0,k]} \right\|), \tag{3.13}$$

where β is a $\mathcal{KL}$-function, γ is a $\mathcal{K}$-function, and $\hat{x}^{\mu}_{0,0}$ is given by the filter initial condition.

Proof. The feasible solution candidate $\hat{\mathbf{v}}^c$, through the standard MPC stability argument, is used to show that $V_N^*(x_k, \hat{\mu}_k)$ is a ISS-Lyapunov function for system (2.1) subject to the control law (3.12)[3].

Firstly, the following bounds hold due to filter stability:

$$\|\tilde{w}_k\| \leq c_{w1} \left\|\mathbf{w}_{[0,k]}\right\|, \quad \|\Delta\hat{\mu}_{k+1}\| \leq c_{w2} \left\|\mathbf{w}_{[0,k]}\right\|, \quad \|\hat{\mu}_k\| \leq c_{w3} \left\|\mathbf{w}_{[0,k]}\right\|,$$

for some $c_{w1}, c_{w2}, c_{w2} > 0 \in \mathbb{R}$. Through Eq. (3.5), we also have:

$$\begin{aligned}\left\|\Delta\hat{x}^{\mu}_{0,k+1}\right\| \leq \rho_{dx}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|), \quad \left\|\hat{x}^{\mu}_{0,k}\right\| \leq \rho_x(c_{w3}\left\|\mathbf{w}_{[0,k]}\right\|),\\ \left\|\Delta\hat{v}^{\mu}_{0,k+1}\right\| \leq \rho_{dv}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|), \quad \left\|\hat{v}^{\mu}_{0,k}\right\| \leq \rho_v(c_{w3}\left\|\mathbf{w}_{[0,k]}\right\|).\end{aligned}$$

Now, the optimal predicted evolution at k and the predicted candidates at $k+1$ can be represented by $x^*(j) = x^*_{k+j|k} - \hat{x}^{\mu}_{0,k}$, and $x^c(j) = x^c_{k+1+j|k+1} - \hat{x}^{\mu}_{0,k+1}$ respectively. Moreover, consider $x^*_{k+N+1|k} = f_\pi(x^*_{k+N|k}, v^*_{k+N|k}) + \hat{\mu}_k$, $x^*(N+1) = x^*_{k+N+1|k} - \hat{x}^{\mu}_{0,k}$, $v^*(j) = v^*_{k+j|k} - \hat{v}^{\mu}_{0,k}$, $v^c(j) = v^c_{k+j+1|k} - \hat{v}^{\mu}_{0,k+1}$, where $v^*_{k+N|k} = v_t(x^*_{k+N|k})$. A modified candidate is defined by $x^c_m(j) = x^c_{k+1+j|k+1} - \hat{x}^{\mu}_{0,k}$, with $x^c_m(j) = x^c(j) + \Delta\hat{x}^{\mu}_{0,k+1}$. The analogous definition holds for $v^c_m(j)$. Since the model is uniform continuous, there exist $\mathcal{K}$-functions $\tilde{\sigma}_j(\cdot)$ such that

$$\left\|x^c_{k+1+j|k+1} - x^*_{k+1+j|k}\right\| \leq \tilde{\sigma}_j(c_{w4}\left\|\mathbf{w}_{[0,k]}\right\|), \ \forall j \in \mathbb{N}_{[0,N]},$$

where $c_{w4} \geq \max(c_{w1}, c_{w2})$, and we define $\sigma_j(\left\|\mathbf{w}_{[0,k]}\right\|) = \tilde{\sigma}_j(c_{w4}\left\|\mathbf{w}_{[0,k]}\right\|)$. Define the following cost variation for notation simplicity:

$$\begin{aligned}\Delta L_\pi(j,x,v) &= L_\pi(x^c(j), v^c(j)) - L_\pi(x^*(j+1), v^*(j+1)),\\ \Delta L^m_\pi(j,x,v) &= L_\pi(x^c_m(j), v^c_m(j)) - L_\pi(x^*(j+1), v^*(j+1)),\\ \Delta V_f(x) &= V_f(x^c(N)) - V_f(x^*(N+1)).\end{aligned}$$

From the uniform continuity of the cost functions (Eqs. (3.10b) and (3.11b)), the following inequalities are thus verified:

$$\begin{aligned}|\Delta L_\pi(j,x,v)| \leq& |\Delta L^m_\pi(j,x,v)| + \lambda_x(\rho_{dx}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|)) + \lambda_v(\rho_{dv}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|))\\ \leq& \lambda_x(\sigma_j(\left\|\mathbf{w}_{[0,k]}\right\|)) + \lambda_x(\rho_{dx}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|)) + \lambda_v(\rho_{dv}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|))\\ =& \xi^j_L(\left\|\mathbf{w}_{[0,k]}\right\|),\\ |\Delta V_f(x)| \leq& \delta(\sigma_N(\left\|\mathbf{w}_{[0,k]}\right\|)) + \delta(\rho_{dx}(c_{w2}\left\|\mathbf{w}_{[0,k]}\right\|))\\ =& \xi_V(\left\|\mathbf{w}_{[0,k]}\right\|),\end{aligned}$$

[3]For notation simplicity, the dependence of V_N^* on $\hat{\mu}_k$ is omitted through this proof.

where ξ_L^j and ξ_V are $\mathcal{K}$-functions. Additionally, from the uniform continuity of $f_\pi(\cdot)$, $v_t(\cdot)$, and $V_f(\cdot)$ and the bounds on $\left\|\hat{x}^\mu_{0,k}\right\|$ and $\|\hat{\mu}_k\|$, the following inequality also holds:

$$\|V_f(x^*(N+1)) - V_f(f_\pi(x^*(N), v_t(x^*(N))))\| \leq \psi(\left\|\mathbf{w}_{[0,k]}\right\|),$$

where ψ is an appropriate $\mathcal{K}$-function.

Therefore, by means of adding and subtracting $\sum_{j=1}^{N} L_\pi(x^*(j), v^*(j)) + V_f(x^*(N+1))$, combined with the bounds of $\Delta L_\pi(j, x, v)$ and $\Delta V_f(x)$, and by using Eq. (3.11c), the candidate cost is bounded by:

$$\begin{aligned}
V_N^c(x_{k+1}) &= \sum_{j=0}^{N-1} L_\pi(x^c(j), v^c(j)) + V_f(x^c(N)) \\
&\leq \sum_{j=1}^{N-1} L_\pi(x^*(j), v^*(j)) + L_\pi(x^*(N), v_t(x^*(N))) + V_f(x^*(N+1)) \\
&\quad + \sum_{j=0}^{N-1} \xi_L^j(\left\|\mathbf{w}_{[0,k]}\right\|) + \xi_V(\left\|\mathbf{w}_{[0,k]}\right\|) \\
&\leq \sum_{j=1}^{N-1} L_\pi(x^*(j), v^*(j)) + V_f(x^*(N)) \\
&\quad + \sum_{j=0}^{N-1} \xi_L^j(\left\|\mathbf{w}_{[0,k]}\right\|) + \xi_V(\left\|\mathbf{w}_{[0,k]}\right\|) + \psi(\left\|\mathbf{w}_{[0,k]}\right\|) \\
&= \sum_{j=0}^{N-1} L_\pi(x^*(j), v^*(j)) + V_f(x^*(N)) \\
&\quad + \theta(\left\|\mathbf{w}_{[0,k]}\right\|) - L_\pi(x^*(0), v^*(0)) \\
&\leq V_N^*(x_k) - \alpha_L(\left\|x_k - \hat{x}^\mu_{0,k}\right\|) + \theta(\left\|\mathbf{w}_{[0,k]}\right\|),
\end{aligned}$$

where $\theta(\left\|\mathbf{w}_{[0,k]}\right\|) = \sum_{j=0}^{N-1} \xi_L^j(\left\|\mathbf{w}_{[0,k]}\right\|) + \xi_V(\left\|\mathbf{w}_{[0,k]}\right\|) + \psi(\left\|\mathbf{w}_{[0,k]}\right\|)$ is a $\mathcal{K}$-function. The property of decreasing cost is thus ensured, since by optimality $V_N^*(x_{k+1}) \leq V_N^c(x_{k+1})$:

$$V_N^*(x_{k+1}) - V_N^*(x_k) \leq -\alpha_L(\left\|x_k - \hat{x}^\mu_{0,k}\right\|) + \theta(\left\|\mathbf{w}_{[0,k]}\right\|). \tag{3.14}$$

Now, notice that $V_N^*(x_k) \geq L_\pi(x^*(0), v^*(0)) \geq \alpha_L(\left\|x_k - \hat{x}^\mu_{0,k}\right\|)$ and, for $x_k \in \mathcal{X}_f$, the unconstrained terminal control law is admissible throughout the entire prediction horizon and by optimality $V_N^*(x_k) \leq V_f(x^*(0)) \leq \beta_V(\left\|x_k - \hat{x}^\mu_{0,k}\right\|)$, $\forall x_k \in \mathcal{X}_f$. Furthermore, from the continuity of the costs and compactness of constraints, $V_N^*(x_k)$ is limited in $\mathcal{X}_N$ and thus, from Lemma B.2, there exists a $\mathcal{K}_\infty$-function $\overline{\beta}_V$ such that:

$$\alpha_L(\left\|x_k - \hat{x}^\mu_{0,k}\right\|) \leq V_N^*(x_k) \leq \overline{\beta}_V(\left\|x_k - \hat{x}^\mu_{0,k}\right\|), \quad \forall x_k \in \mathcal{X}_N \tag{3.15}$$

Finally, from (3.14) and (3.15), considering the modified state $\tilde{x}_k = x_k - \hat{x}^\mu_{0,k}$, $V_N^*(x_k)$ is an ISS-Lyapunov function and, through Lemma B.1, Eq. (3.13) is satisfied. $\square$

Therefore, this NMPC control algorithm formally incorporates the mean-value disturbance estimates and steady-state correction into the prediction model and optimization problem, maintaining recursive feasibility and ISS-stability guarantees. The main characteristics of this controller, in particular the offset correction provided in the presence of constant disturbances, are further illustrated through case-studies in Chapter 7.

Recapitulation

In this chapter, the problem of steady-state *offset* in the presence of constant disturbances was tackled. The NMPC presented in Chapter 2 was modified in order to avoid this undesired effect, via incorporating a constant disturbance model into the prediction and correcting the target. In particular, the following topics were discussed:

- Reachable equilibrium in the presence of constant disturbances: It was shown how a modified equilibrium can be defined such that the equilibrium condition is satisfied in the presence of a constant disturbance, while the output is still steered to its desired value. The conditions under which this equilibrium point is uniquely defined by the constant disturbance were also presented.

- Constant disturbance estimation and disturbance propagation: The process of estimating the disturbance mean-value via filtration of the measured additive disturbance was presented, and the disturbance propagation condition discussed in Chapter 2 was generalized in order to incorporate the estimated disturbance means and their actualization.

- Robust NMPC with constant disturbance attenuation: The model predictive controller with the incorporation of the estimated disturbance mean into the prediction model and target correction was presented. Under similar assumptions as in the nominal prediction case, it was shown that recursive feasibility and input-to-state stability are still guaranteed.

Chapter 4

Zonotopic Uncertainty Propagation

In this Chapter, algorithms for the computation of disturbance propagation sets $\mathcal{S}(j)$ based on the zonotopic mean-value extension is presented. As mentioned in Section 2.3, for linear systems it is possible to compute the smallest sets $\mathcal{S}^*(j)$. In the nonlinear case, however, more conservative outer bounds must be considered.

A simple disturbance propagation method for nonlinear systems uses Lipschitz constants [19] and is described in Section 4.1. However, the resulting sets tend to be rather conservative, since this approach propagates the worst-case gain identically in all directions. The method via zonotopes proposed in Section 4.2 does not present this sort of conservatism.

It is then proven that the proposed approach results in smaller sets $\mathcal{S}(j)$ than the ones obtained by the Lipschitz infinity-norm and thus less conservative tightened constraints $\mathcal{Z}_\pi(j)$, as well as a potentially larger domain of attraction. Finally, a natural extension to constrained zonotopes [35] is presented.

4.1 Method via Lipschitz Constants

Given $L_x \in \mathbb{R}$ a Lipschitz constant for f_π in $\mathcal{Z}_\pi$, i.e. $\|f_\pi(x_b, v) - f_\pi(x_a, v)\| \leq L_x \|x_b - x_a\|$ for any (x_a, v), $(x_b, v) \in \mathcal{Z}_\pi$, and $\mathcal{S}_l(0) = \{x \in \mathbb{R}^n : \|x\| \leq w_m\} \supseteq \mathcal{W}$, sets $\mathcal{S}_l(j)$ that satisfy Condition 2.1 can be given by

$$\mathcal{S}_l(j) = \{w \in \mathbb{R}^n : \|w\| \leq L_x^j w_m\}, \quad j = 0 \dots N. \tag{4.1}$$

If a constant disturbance model is incorporated into the prediction, the more general Condition 3.1 must be considered. With this in mind, if $\overline{w}_m, \delta_m \in \mathbb{R}$ are such that $\overline{\mathcal{W}} \subseteq \{w \in \mathbb{R}^n : \|w\| \leq \overline{w}_m\}$ and $\mathcal{DM} \subseteq \{\delta \in \mathbb{R}^n : \|\delta\| \leq \delta_m\}$, sets $\mathcal{S}_l^\mu(j)$ satisfying Condition 3.1 are given by

$$\mathcal{S}_l^\mu(j) = \{w \in \mathbb{R}^n \colon \|w\| \leq w_L(j)\}, \quad j = 0 \dots N, \tag{4.2}$$

where $w_L(j) = L_x^j \overline{w}_m + \left(\sum_{i=0}^{j-1} L_x^i\right) \delta_m$ [1]. Notice that each set $\mathcal{S}_l^\mu(j)$ is defined by the scalar $w_L(j) \in \mathbb{R}$, with $w_L(j+1) = L_x w_L(j) + \delta_m$. The worst case gain, represented by the Lipschitz constant, is thus propagated equally in all directions and any asymmetries of the model function f_π are lost.

Another idea for computing disturbance propagation sets would be using interval analysis to propagate box constraints through the system dynamics. Indeed, given the interval set $\mathcal{S} = \{(x, v) \in \mathbb{R}^{n \times m} \colon x_j^{min} \leq x_j \leq x_j^{max},\ v_i^{min} \leq v_i \leq v_i^{max},\ j = 1 \dots n,\ i = 1 \dots m\}$, interval extensions could be used to obtain an outer approximation of $f_\pi(\mathcal{S})$.

However, in order to obtain sets $\mathcal{S}(j)$ satisfying Conditions 2.1 or 3.1, the image of $(x_a, v) \oplus (\mathcal{S}(j) \times \{0\})$ needs to be considered, where $(x_a, v) \in \mathcal{Z}_\pi(j-1)$ is not known *a priori*. This means that interval analysis is not applicable, since even if $\mathcal{Z}_\pi(j-1)$ and $\mathcal{S}(j)$ are interval sets, the set $\{(x_a, v, x_b) \in \mathcal{Z}_\pi(j-1) \times \mathbb{R}^n \colon x_b - x_a \in \mathcal{S}(j)\}$ is not an interval, and thus an interval outer approximation of $f_\pi(x_b, v) - f_\pi(x_a, v)$ cannot in general be obtained through interval analysis methods.

In the next section, a zonotopic disturbance propagation method, which is less conservative than the Lipschitz method and can directly deal with any $(x_a, v) \in \mathcal{Z}_\pi(j-1)$, is proposed.

4.2 Zonotopic Method

In order to calculate disturbance propagation sets which satisfy Conditions 2.1 and 3.1 using zonotopes, an algorithm for obtaining an outer bound for the image of a zonotope $X = \{G, c\} \subseteq \mathbb{R}^m$ by a nonlinear function $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n$ is necessary. In particular, we need a zonotope $Y \subseteq \mathbb{R}^n$ that satisfies $\varphi(X) \subseteq Y$.

Lemma 4.1 [1, 31] allows the computation of a *zonotopic extension* of the product of a centered zonotope by an interval matrix.

Lemma 4.1 ([1]). *Given a centered zonotope $X = M\mathcal{B}_\infty^{n_g} \subseteq \mathbb{R}^m$ and an interval matrix $\mathbf{J} \in \mathbb{I}^{n \times m}$, consider the zonotope family $\mathbf{Z} = \mathbf{J}X = \{Jx \colon J \in \mathbf{J},\ x \in X\}$. A zonotopic inclusion $\diamond(\mathbf{Z})$ is defined as*

$$\diamond(\mathbf{Z}) = mid(\mathbf{J})X \oplus P\mathcal{B}_\infty^n, \tag{4.3}$$

where P is a diagonal matrix satisfying

$$P_{ii} = \sum_{j=1}^{n_g} \sum_{k=1}^{m} rad(\mathbf{J})_{ik} |M_{kj}|, \quad i = 1 \dots n. \tag{4.4}$$

From these definitions, we have $\mathbf{Z} \subseteq \diamond(\mathbf{Z})$.

[1] If $\mathcal{M} = \{0\}$ ($\delta_m = 0$, $\overline{w}_m = w_m$), then $w_L(j) = L_x^j w_m$ and Eq. (4.1) is recovered.

Proof. Given $x \in X$ and $J \in \mathbf{J}$, we have $x = M\xi$ and $J_{ik} = mid(\mathbf{J})_{ik} + rad(\mathbf{J})_{ik}\psi_{ik}$, where $\xi_j \in [-1, 1]$, $j = 1 \ldots n_g$, $\psi_{ik} \in [-1, 1]$, $i = 1 \ldots n, j = 1 \ldots m$. Therefore, for any $z = Jx \in \mathbf{Z}$, we have

$$(z - mid(\mathbf{J})M\xi)_i = \sum_{j=1}^{n_g} \sum_{k=1}^{m} rad(\mathbf{J})_{ik} M_{kj} \psi_{ik} \xi_j$$

and, since $|\psi_{ik}\xi_j| \leq 1$, $|(z - mid(\mathbf{J})M\xi)_i| \leq \sum_{j=1}^{n_g} \sum_{k=1}^{m} rad(\mathbf{J})_{ik}|M_{kj}| = P_{ii}$ and $(z - mid(\mathbf{J})M\xi) \in P\mathcal{B}_\infty^n$. Therefore, we have $z \in mid(\mathbf{J})X \oplus P\mathcal{B}_\infty^n = \diamond(\mathbf{Z})$. □

Based on Lemma 4.1 and the mean-value theorem, Theorem 4.1 [1, 31] defines the *mean-value extension* of zonotopes.

Theorem 4.1 (Mean-value zonotopic extension [1]). *Given $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n$ a class $\mathcal{C}^1$ function, $X = h \oplus M\mathcal{B}_\infty^{n_g} \subseteq \mathbb{R}^m$ a zonotope and $\mathbf{J} \in \mathbb{I}^{n \times m}$ an interval matrix satisfying $\nabla^\intercal\varphi(X) \subseteq \mathbf{J}$, we have*

$$\begin{aligned} \varphi(X) &\subseteq \varphi(h) \oplus \diamond(\mathbf{J}(X - h)) \\ &= \varphi(h) \oplus \begin{pmatrix} mid(\mathbf{J})M & P \end{pmatrix} \mathcal{B}_\infty^{n_g+n}, \end{aligned} \tag{4.5}$$

where P is defined as in (4.4).

Proof. From the application of the mean-value theorem, given $y \in \varphi(x)$, with $x \in X$, there is a $J \in \mathbf{J}$ such that:

$$y = \varphi(h) + J(x - h).$$

Then, from Lemma 4.1, we have $J(x - h) \in \diamond(\mathbf{J}(X - h))$ and thus $y = \varphi(h) + J(x - h) \in \varphi(h) \oplus \diamond(\mathbf{J}(X - h))$. □

Based on the mean-value extension of Theorem 4.1, an algorithm for the iterative computation of zonotopes $\mathcal{S}_z(j) \subseteq \mathbb{R}^n$ satisfying Condition 2.1 can be developed.

Property 4.1. *Consider the nonlinear system with additive disturbances* (2.4a) *and let $\mathbf{J}_\pi \in \mathbb{I}^{n \times n}$ be an interval matrix satisfying $\nabla_x^\intercal f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$. Consider the zonotopes $\mathcal{S}_z(j)$, $j = 0 \ldots N$ defined by*

(i) *$\mathcal{S}_z(0)$ is a centered zonotope which contains $\mathcal{W}$.*

(ii) *$\mathcal{S}_z(j) = \diamond(\mathbf{J}_\pi \mathcal{S}_z(j-1))$, $\quad j = 1 \ldots N$.*

These sets satisfy Condition 2.1.

Proof. The condition of $\mathcal{S}_z(0)$ compact with $\mathcal{W} \subseteq \mathcal{S}_z(0)$ is satisfied by design. For each $j = 1 \ldots N$ and given any $x_a \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, with $(x_a, v) \oplus (\mathcal{S}_z(j-1) \times \{0\}) \subseteq \mathcal{Z}_\pi$, consider the function $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\varphi(x) = f_\pi(x, v)$.

From $X_a = x_a \oplus \mathcal{S}_z(j-1)$, with $X_a \times \{v\} \subseteq \mathcal{Z}_\pi$, we have $\nabla^\intercal \varphi(X_a) \subseteq \mathbf{J}_\pi$. Thus, from Theorem 4.1,

$$
\begin{aligned}
f_\pi(X_a, v) &= \varphi(X_a) \subseteq \varphi(x_a) \oplus \diamond(\mathbf{J}_\pi \mathcal{S}_z(j-1)) \\
f_\pi(X_a, v) &\subseteq f_\pi(x_a, v) \oplus \mathcal{S}_z(j)
\end{aligned}
$$

and for any $x_b \in X_a$, $f_\pi(x_b, v) \in f_\pi(x_a, v) \oplus \mathcal{S}_z(j)$. □

The sets $\mathcal{S}_z(j)$ given by Property 4.1 can, therefore, be applied in the constraint tightening approach described in (2.8). Notice that, unlike the Lipschitz method, this algorithm considers the form of the nonlinear function f_π through the interval matrix $\mathbf{J}_\pi$.

An interval matrix $\mathbf{J}_\pi$ satisfying $\nabla_x^\intercal f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$ can be directly obtained from $\mathbb{I}(\mathcal{Z}_\pi)$ by means of interval arithmetic [28]. Alternatively, if $\mathbf{J}_x \in \mathbb{I}^{n\times n}$ and $\mathbf{J}_u \in \mathbb{I}^{n\times m}$ are such that $\nabla_x^\intercal f(\mathcal{Z}_\pi) \subseteq \mathbf{J}_x$ and $\nabla_u^\intercal f(\mathcal{Z}_\pi) \subseteq \mathbf{J}_u$, we have

$$
\mathbf{J}_\pi = \mathbf{J}_x + \mathbf{J}_u K_v, \tag{4.6}
$$

where the matrix sums and products are made through interval arithmetic [28]. Equation (4.6) emphasizes the effect of the feedback matrix K_v on $\mathbf{J}_\pi$ and, consequently, on the sets $\mathcal{S}_z(j)$, and can be used to choose a matrix K_v in order to reduce the disturbance propagation, as proposed in Appendix A.

Remark 4.1. *Due to the zonotopic inclusion, the number of generators of the zonotopes $\mathcal{S}_z(j)$ increases for each iteration. Methods for complexity reduction [10, 35], such as the one described in Appendix A, can be used to restrict the number of generators of each $\mathcal{S}_z(j)$ to a predefined value.*

Remark 4.2. *The Pontryagin difference of a polytope by a zonotope can be made algebraically with low computational cost (Appendix A). This simplifies the constraint tightening process for the case of polytopic constraints $\mathcal{Z}_\pi$.*

For the constant disturbance attenuation case, the changes on the mean-value estimate $\hat{\mu}$ influence the one-step-ahead disturbance propagation, since the sequence of future disturbances $\hat{\mathbf{w}}$ considered is potentially different at the instants k and $k+1$. This effect can be taken into consideration by adding the set $\mathcal{DM}$ at each iterative step in the definition of the $\mathcal{S}_z(j)$, as detailed in Property 4.2.

Property 4.2. *Consider system* (2.4a)*, a centered zonotope $\overline{\mathcal{DM}} \subseteq \mathbb{R}^n$ and an interval matrix $\mathbf{J}_\pi \in \mathbb{I}^{n\times n}$, with $\mathcal{DM} \subseteq \overline{\mathcal{DM}}$ and $\nabla_x^\intercal f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$. Consider the sets $\mathcal{S}_z^\mu(j)$, $j = 0 \ldots N$ defined by* *(i)* $\mathcal{S}_z^\mu(0)$ *is a centered zonotope that contains* $\overline{\mathcal{W}}$*.*

(ii) $\mathcal{S}_z^\mu(j) = \diamond(\mathbf{J}_\pi \mathcal{S}_z^\mu(j-1)) \oplus \overline{\mathcal{DM}}, \quad j = 1 \ldots N.$

Such zonotopes satisfy Condition 3.1.

Proof. The first condition is satisfied by design. Considering $x_a, x_b \in \mathbb{R}^n$, $v \in \mathbb{R}^{n_u}$ and $\mu_a, \mu_b \in \mathcal{M}$ as given in the second condition for some $j = 1 \ldots N$, and $\Delta_j = (f_\pi(x_b, v) + \mu_b) - (f_\pi(x_a, v) + \mu_a)$, since $x_b - x_a \in \mathcal{S}_z^\mu(j-1)$, we have

$$\begin{aligned}
\Delta_j &= (f_\pi(x_b, v) - f_\pi(x_a, v)) + (\mu_b - \mu_a) \\
&\in \mathbf{J}_\pi \mathcal{S}_z^\mu(j-1) + (\mu_b - \mu_a) \\
&\subseteq \diamond(\mathbf{J}_\pi \mathcal{S}_z^\mu(j-1)) \oplus \overline{\mathcal{DM}} = \mathcal{S}_z^\mu(j).
\end{aligned}$$

Therefore, the sets $\mathcal{S}_z^\mu(j)$ satisfy Condition 3.1. □

Notice that Property 4.1 can be seen as a particular case of Property 4.2 for the nominal prediction case ($\mathcal{DM} = \{0\}$, $\overline{\mathcal{W}} = \mathcal{W}$). The size of the sets $\mathcal{S}_z^\mu(j)$ is highly dependent on the set $\mathcal{DM}$. Indeed, $\hat{\mu}_k$ is used to estimate the constant steady-state disturbance condition, but transient effects, and thus changes in the prediction model, are taken into account in the set $\mathcal{DM}$. As discussed in Section 3.2, if necessary $\mathcal{DM}$ can be significantly reduced from the definition of the constant disturbance estimates $\hat{\mu}_k$.

Notice that, in the recursive construction of the sets $\mathcal{S}_z^\mu(j)$ proposed in Property 4.2, the set $\overline{\mathcal{DM}}$ is added at each step, which amounts to considering a potentially different $\Delta\hat{\mu}(j) \in \mathcal{DM}$ for each $j = 1 \ldots N$. This comes from Condition 3.1, which considers that $(f_\pi(x_b, v) + \mu_b) - (f_\pi(x_a, v) + \mu_a) \in \mathcal{S}(j)$ must be satisfied if $x_b - x_a \in \mathcal{S}(j-1)$, for any $\mu_a, \mu_b \in \mathcal{M}$, $\mu_b - \mu_a \in \mathcal{DM}$, where the pair μ_a, μ_b may be different for each $j = 1 \ldots N$.

However, the sets $\mathcal{S}(j)$ are only required to limit the differences $x_{k+1+j|k+1} - x_{k+1+j|k}$, $j = 0 \ldots N$, and the predicted trajectories $(x_{k+1+j|k}, x_{k+1+j|k+1})$ are obtained from the same pair of disturbance estimates $(\hat{\mu}_k, \hat{\mu}_{k+1})$ for every $j = 0 \ldots N$. Therefore, considering a potentially different pair of disturbance estimates μ_a, μ_b at each iterative step, as done in Condition 3.1, can be conservative.

Taking this aspect into consideration, Property 4.3 provides an alternative algorithm to compute zonotopes $\mathcal{S}_z(j)$ which limit the one-step-ahead disturbance propagation when a constant disturbance model is incorporated, which does not suffer from this source of conservatism.

Property 4.3. *Given the system* (2.4a)*, a zonotope* $\overline{\mathcal{DM}} \subseteq \mathbb{R}^n$ *and an interval matrix* $\mathbf{J}_\pi \in \mathbb{I}^{n \times n}$*, with* $\mathcal{DM} \subseteq \overline{\mathcal{DM}}$ *and* $\nabla_x^\intercal f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$*, consider the sets* $\overline{\mathcal{S}_z^\mu}(j)$*,* $j = 0 \ldots N$ *defined by*

$$\overline{\mathcal{S}_z^\mu}(j) = \mathcal{S}^0(j) \oplus \diamond(\mathbf{T}_j \overline{\mathcal{DM}}),$$

where $\mathcal{S}^0(0)$ is a zonotope that contains $\overline{\mathcal{W}}$, $\mathcal{S}^0(j+1) = \diamond(\mathbf{J}_\pi \mathcal{S}^0(j))$ and $\mathbf{T}_0 = \mathbf{0}$, $\mathbf{T}_{j+1} = \mathbf{I} + \mathbf{J}_\pi \mathbf{T}_j$.[2] *These zonotopes satisfy $x_{k+j|k+1} \in x_{k+j|k} \oplus \overline{\mathcal{S}_z^\mu}(j-1),\ j = 1 \dots N+1$ for predictions (3.7) and can thus also be used for constraint tightening purposes in the constant disturbance attenuation case.*

Proof. It will be shown that $x_{k+1+j|k+1} - x_{k+1+j|k} = s_j + T_j \Delta\hat{\mu}_{k+1},\ j = 0 \dots N$, for some $s_j \in \mathcal{S}^0(j)$ and $T_j \in \mathbf{T}_j$. First, notice that $x_{k+1} - x_{k+1|k} = w_k - \hat{\mu}_k = s_0$, where $s_0 \in \overline{\mathcal{W}} \subseteq \mathcal{S}^0(j)$ (we have $T_0 \Delta\hat{\mu}_{k+1} = 0$, since $\mathbf{T}_0 = \mathbf{0}$). Now, considering by induction that $x_{k+j|k+1} - x_{k+j|k} = s_{j-1} + T_{j-1}\Delta\hat{\mu}_{k+1}$, with $s_{j-1} \in \mathcal{S}^0(j-1)$, $T_{j-1} \in \mathbf{T}_{j-1}$, there exists $J \in \mathbf{J}_\pi$ such that

$$\begin{aligned} x_{k+1+j|k+1} - x_{k+1+j|k} &= f_\pi(x_{k+j|k+1}, v_j) + \hat{\mu}_{k+1} - (f_\pi(x_{k+j|k}, v_j) + \hat{\mu}_k) \\ &= J(s_{j-1} + T_{j-1}\Delta\hat{\mu}_{k+1}) + \Delta\hat{\mu}_{k+1} \\ &= Js_{j-1} + (JT_{j-1} + I)\Delta\hat{\mu}_{k+1} \\ &= s_j + T_j \Delta\hat{\mu}_{k+1}, \end{aligned}$$

where $s_j = Js_{j-1} \in \mathcal{S}^0(j)$ and $T_j = I + JT_{j-1} \in \mathbf{T}_j$. Finally, since $\Delta\hat{\mu}_{k+1} \in \mathcal{DM} \subseteq \overline{\mathcal{DM}}$, we have $x_{k+1+j|k+1} - x_{k+1+j|k} = s_j + T_j \Delta\hat{\mu}_{k+1} \in \mathcal{S}^0(j) \oplus \diamond(\mathbf{T}_j \overline{\mathcal{DM}})$. □

Remark 4.3. *Notice that both zonotopes $\mathcal{S}_z(j)$ and $\overline{\mathcal{S}_z^\mu}(j)$ obtained from Properties 4.1 and 4.3 are reduced in the linear case ($f_\pi(x,v) = A_v x + Bv$) to the optimal disturbance propagation sets: $\mathcal{S}^*(j) = A_v^j \mathcal{W}$ and $\mathcal{S}^*(j) = A_v^j \overline{\mathcal{W}} \oplus \left(\sum_{i=0}^{j-1} A_v^i\right) \mathcal{DM}$ in the nominal and constant disturbance model cases, respectively.*

4.3 Comparison of Criteria

In this section, the conservatism reduction brought by the zonotopic methods is formalized by showing that the zonotopic disturbance propagation sets are contained in the ones computed by the Lipschitz infinity-norm method. The infinity-norm is considered in order to simplify comparisons between both approaches, since in this case the $\mathcal{S}_l(j)$ are boxes and, therefore, also zonotopes.

For the calculation of an infinity-norm Lipschitz constant for the system dynamics function $f_\pi \colon \mathcal{Z}_\pi \to \mathbb{R}^n$, the following theorem from multivariable calculus [18] can be applied.

Theorem 4.2 ([18]). *Given a function $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$, a convex set $X \subseteq \mathbb{R}^n$ and a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, a real number $L > 0$ is a Lipschitz constant for φ in X, that*

[2] $\mathbf{I}$ and $\mathbf{0}$ represent here the degenerate interval matrices consisting, respectively, of only the identity and null matrices of appropriate dimensions.

is

$$\|\varphi(x_b) - \varphi(x_a)\| \leq L\,\|x_b - x_a\|\,, \quad \forall\, x_a, x_b \in X, \tag{4.7}$$

if and only if the jacobian $\nabla^{\intercal}\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *satisfies*

$$\|\nabla^{\intercal}\varphi(x)\| \leq L, \quad \forall\, x \in X, \tag{4.8}$$

where $\|\cdot\|$ *in* (4.8) *represents the induced norm of the linear transformation.*

Therefore, being $\mathbf{J}_\pi \in \mathbb{I}^{n\times n}$ an interval matrix satisfying $\nabla_x^{\intercal} f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$, an infinity-norm Lipschitz constant for f_π in $\mathcal{Z}_\pi$ is given by

$$L_x = \max_{J\in\mathbf{J}_\pi}\ \|J\|_\infty\,, \tag{4.9}$$

that is, we have $\|f_\pi(x_b, v) - f_\pi(x_a, v)\|_\infty \leq L_x\,\|x_b - x_a\|_\infty$ for all $(x_a, v), (x_b, v) \in \mathcal{Z}_\pi$. Based on this relationship between $\mathbf{J}_\pi$ and L_x, Theorem 4.3 and Corolary 4.1 prove that the zonotopic disturbance propagation sets can be easily defined to be contained in the respective Lipschitz sets.

Theorem 4.3. *Consider system* (2.4a) *and let* $\mathbf{J}_\pi \in \mathbb{I}^{n\times n}$ *be an interval matrix satisfying* $\nabla_x^{\intercal} f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$. *Let* $\mathcal{S}_z^\mu(j)$ *and* $\overline{\mathcal{S}_z^\mu}(j)$ *be zonotopes obtained by the methods proposed in Properties 4.2 and 4.3, respectively, and* $\mathcal{S}_l^\mu(j)$ *be boxes defined by Eq.* (4.2), *with* $L_x = \max_{J\in\mathbf{J}_\pi}\ \|J\|_\infty$. *Assuming* $\mathcal{S}_z^\mu(0), \overline{\mathcal{S}_z^\mu}(0) \subseteq \mathcal{S}_l^\mu(0)$ *and* $\overline{\mathcal{DM}} \subseteq \delta_m\mathcal{B}_\infty^n$, *we have* $\mathcal{S}_z^\mu(j), \overline{\mathcal{S}_z^\mu}(j) \subseteq \mathcal{S}_l^\mu(j)$ *for all* $j = 0\ldots N$. [3]

Proof. This proof will be separated into two parts, the first corresponding to the affirmation $\mathcal{S}_z^\mu(j) \subseteq \mathcal{S}_l^\mu(j)$ and the second to $\overline{\mathcal{S}^\mu}(j) \subseteq \mathcal{S}_l^\mu(j)$. The notation $\iota(A)$ introduced in Appendix C to represent the diagonal matrix whose elements are the sums of the lines of the matrix A will also be used.

(i) Since $\mathcal{S}_z^\mu(0) \subseteq \mathcal{S}_l(0)$, it is sufficient by induction to show that $\mathcal{S}_z^\mu(j) \subseteq \mathcal{S}_l^\mu(j)$ implies $\mathcal{S}_z^\mu(j+1) \subseteq \mathcal{S}_l(j+1)$ for all $j = 0\ldots N-1$.

Given $\mathcal{S}_z^\mu(j) = M\mathcal{B}_\infty^{n_g}$, we have

$$\begin{aligned}\mathcal{S}_z(j+1) &= \diamond(\mathbf{J}_\pi\mathcal{S}_z(j)) \oplus \overline{\mathcal{DM}}\\ &\subseteq \Big(mid(\mathbf{J}_\pi)M \quad \iota(rad(\mathbf{J}_\pi)|M|) \quad \delta_m I\Big)\,\mathcal{B}_\infty^{n_g+2n}.\end{aligned}$$

Defining $\overline{M} = \Big(mid(\mathbf{J}_\pi)M \quad \iota(rad(\mathbf{J}_\pi)|M|) \quad \delta_m I\Big)$, we have

$$\begin{aligned}\|\overline{M}\|_\infty &= \max_i\,(\iota(|mid(\mathbf{J}_\pi)M|)_{ii} + \iota(rad(\mathbf{J}_\pi)|M|)_{ii} + \delta_m)\\ &\leq \max_i\,(\iota(J^*|M|)_{ii}) + \delta_m\\ &= \|J^*|M|\|_\infty + \delta_m,\end{aligned}$$

[3] $A, B \subseteq C$ is used here as a compact way of representing $A \subseteq C$ and $B \subseteq C$.

where $J^* = |mid(\mathbf{J}_\pi)| + rad(\mathbf{J}_\pi)$. Notice that, from the induction hypothesis $\mathcal{S}_z^\mu(j) \subseteq \mathcal{S}_l^\mu(j)$, $\|M\|_\infty \leq w_L(j)$. We also have $\|J^*\|_\infty = \max_{J\in\mathbf{J}_\pi} \|J\|_\infty = L_x$, therefore,

$$\begin{aligned}\left\|\overline{M}\right\|_\infty &\leq \|J^*|M|\|_\infty + \delta_m \\ &\leq \|J^*\|_\infty \|M\|_\infty + \delta_m \\ &\leq L_x w_L(j) + \delta_m = w_L(j+1),\end{aligned}$$

which corresponds to $\mathcal{S}_z^\mu(j+1) = \overline{M}\mathcal{B}_\infty^{n_g+2n} \subseteq \mathcal{S}_l(j+1)$.

(ii) Based on the previous proof, considering the particular case of $\overline{\mathcal{DM}} = \{0\}$ and $\delta_m = 0$, we have $\mathcal{S}^0(j) \subseteq L_x^j \overline{w}_m \mathcal{B}_\infty^n$, $j = 0\dots N$. Therefore, it suffices to show that $\diamond(\mathbf{T}_j\overline{\mathcal{DM}}) \subseteq (\sum_{i=0}^{j-1} L_x^i)\delta_m\mathcal{B}_\infty^n$, $\forall j = 0\dots N$.

Notice that, from induction, $\|\mathbf{T}_j\|_\infty \leq \sum_{i=0}^{j-1} L_x^i$, $\forall j = 0\dots N$, since the equality is trivially valid for $j = 0$ and $\|\mathbf{T}_{j+1}\|_\infty \leq \|\mathbf{I}\|_\infty + \|\mathbf{J}_\pi\|_\infty \|\mathbf{T}_j\|_\infty = 1 + L_x \|\mathbf{T}_j\|_\infty$.[4]

Therefore, being $\overline{\mathcal{DM}} = M\mathcal{B}_\infty^{n_g}$, we have

$$\begin{aligned}\diamond(\mathbf{T}_j\overline{\mathcal{DM}}) &= \begin{pmatrix} mid(\mathbf{T}_j)M & \iota(rad(\mathbf{T}_j)|M|)\end{pmatrix}\mathcal{B}_\infty^{n_g+n} \\ &\subseteq \begin{pmatrix} \iota(|mid(\mathbf{T}_j)M|) & \iota(rad(\mathbf{T}_j)|M|)\end{pmatrix}\mathcal{B}_\infty^{2n} \\ &\subseteq \left\|T_j^*|M|\right\|_\infty \mathcal{B}_\infty^n,\end{aligned}$$

where $T_j^* = |mid(\mathbf{T}_j)| + rad(\mathbf{T}_j)$. Finally, since $\left\|T_j^*\right\|_\infty = \|\mathbf{T}_j\|_\infty \leq \sum_{i=0}^{j-1} L_x^i$ and, from $\overline{\mathcal{DM}} \subseteq \delta_m\mathcal{B}_\infty^n$, $\|M\|_\infty \leq \delta_m$, $\left\|T_j^*|M|\right\|_\infty \leq \left\|T_j^*\right\|_\infty \|M\|_\infty \leq (\sum_{i=0}^{j-1} L_x^i)\delta_m$ and $\diamond(\mathbf{T}_j\overline{\mathcal{DM}}) \subseteq (\sum_{i=0}^{j-1} L_x^i)\delta_m\mathcal{B}_\infty^n$ follows.

□

The restrictions $\mathcal{S}_z^\mu(0), \overline{\mathcal{S}_z^\mu}(0) \subseteq \mathcal{S}_l(0)$ and $\overline{\mathcal{DM}} \subseteq \delta_m\mathcal{B}_\infty^n$ can be trivially satisfied by making $\mathcal{S}_z^\mu(0) = \overline{\mathcal{S}_z^\mu}(0) = \mathcal{S}_l(0)$ and $\overline{\mathcal{DM}} = \delta_m\mathcal{B}_\infty^n$, since every box is a zonotope. The liberty of considering any zonotopes as $\mathcal{S}_z(0)$, $\overline{\mathcal{S}_z^\mu}(0)$ and $\overline{\mathcal{DM}}$ can provide still another source of conservatism reduction.

Corolary 4.1. *Consider system* (2.4a) *and let* $\mathbf{J}_\pi \in \mathbb{I}^{n\times n}$ *be an interval matrix satisfying* $\nabla_x^\intercal f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$. *Let* $\mathcal{S}_z(j)$ *be zonotopes computed through the method proposed in Property 4.1 and* $\mathcal{S}_l(j) = L_x^j w_m \mathcal{B}_\infty^n$ *be boxes, with* $L_x = \max_{J\in\mathbf{J}_\pi} \|J\|_\infty$ *and* $\mathcal{W} \subseteq \mathcal{S}_l(0) = w_m\mathcal{B}_\infty^n$. *Assuming* $\mathcal{S}_z(0) \subseteq \mathcal{S}_l(0)$, *we have* $\mathcal{S}_z(j) \subseteq \mathcal{S}_l(j)$ *for all* $j = 0\dots N$.

Proof. Follows from Theorem 4.3 by making $\overline{\mathcal{DM}} = \{0\}$, $\overline{\mathcal{W}} = \mathcal{W}$ and $\delta_m = 0$. □

[4]The infinity norm of an interval matrix $\mathbf{A} \in \mathbb{I}^{m\times n}$ is used here to compactly represent $\|\mathbf{A}\|_\infty = max_{A\in\mathbf{A}} \|A\|_\infty$.

4.4 Extension to Constrained Zonotopes

The disturbance propagation via zonotopes proposed in Section 4.2 requires zonotopic outer approximations of the additive disturbance set $\mathcal{W}$ and, for the constant disturbance attenuation case, of the mean-value estimate actualization set $\mathcal{DM}$. However, due to the inherent symmetry of zonotopes, these approximations, and the sets $\mathcal{S}(j)$ that ensue, can be rather conservative in case the sets $\mathcal{W}$ and $\mathcal{DM}$ are asymmetrical. This outer approximation problem can be mitigated by considering constrained zonotopes (CZ), which are not necessarily symmetrical and do not suffer from this source of conservatism.[5]

Constrained zonotopes [35, 31] are an extension of zonotopes, considering the presence of linear constraints on the generators. A constrained zonotoped is defined by:

$$Z = \{z = c + G\xi \colon \ \|\xi\|_\infty \leq 1, \ A\xi = b\}, \tag{4.10}$$

where, as in zonotopes, $c \in \mathbb{R}^n$ and $G \in \mathbb{R}^{n \times n_g}$ are the center and generator matrix of the constrained zonotope. The matrix $A \in \mathbb{R}^{n_c \times n_g}$ and vector $b \in \mathbb{R}^{n_c}$ represent the linear constraints on the generators, which are assumed without loss of generality to be independent (A is a full-line rank matrix). For simplicity of notation, we use

$$Z = c \oplus G\mathcal{B}_\infty(A, b) = \{G, c, A, b\}, \tag{4.11}$$

where $\mathcal{B}_\infty(A, b) = \{\xi \in \mathbb{R}^{n_g} \colon \ \|\xi\|_\infty \leq 1, \ A\xi = b\}$ is a constrained unitary box. The Minkowski sum and linear transformation of constrained zonotopes can also be made algebraically with low computational cost. Given $Z_1 = \{G_1, c_1, A_1, b_1\}$, $Z_1 = \{G_2, c_2, A_2, b_2\} \in \mathbb{R}^n$ and $R \in \mathbb{R}^{m \times n}$, we have:

$$RZ_1 = \{RG_1, Rc_1, A, b\}, \tag{4.12}$$

$$Z_1 \oplus Z_2 = \left\{ \begin{pmatrix} G_1 & G_2 \end{pmatrix}, c_1 + c_2, \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\}. \tag{4.13}$$

The mean-value extension of zonotopes presented in Section 4.2 can then be extended for constrained zonotopes, as shown by Lemmas 4.2 and 4.3 [35, 31].

Lemma 4.2 ([31]). *Given a centered constrained zonotope $X = M\mathcal{B}_\infty(A, b) \subseteq \mathbb{R}^m$, an interval matrix $\mathbf{J} \in \mathbb{I}^{n \times m}$ and a zonotope $\overline{X} = \overline{M}\mathcal{B}_\infty^{n_g} \supseteq X$, consider the set $\mathbf{Z} = \mathbf{J}X = \{Jx \colon \ J \in \mathbf{J}, \ x \in X\}$. A CZ-inclusion $\triangleleft(\mathbf{Z})$ is defined as*

$$\triangleleft(\mathbf{Z}) = mid(\mathbf{J})X \oplus P\mathcal{B}_\infty^n, \tag{4.14}$$

where P is a diagonal matrix satisfying

$$P_{ii} = \sum_{j=1}^{n_g} \sum_{k=1}^{m} rad(\mathbf{J})_{ik} |\overline{M}_{kj}|, \quad i = 1 \dots n. \tag{4.15}$$

[5]In fact, as shown in [35], any convex, compact polytope can be represented as a constrained zonotope.

From these definitions, we have $\mathbf{Z} \subseteq \triangleleft(\mathbf{Z})$.

Proof. This proof follows the same arguments as that of Lemma 4.1. Given $x \in X \subseteq \overline{X}$ and $J \in \mathbf{J}$, we have $x = \overline{M}\xi$ and $J_{ik} = mid(\mathbf{J})_{ik} + rad(\mathbf{J})_{ik}\psi_{ik}$, where $\xi_j \in [-1,1],\ j = 1 \dots n_g$, $\psi_{ik} \in [-1,1],\ i = 1 \dots n, j = 1 \dots m$. Therefore, for any $z = Jx \in \mathbf{Z}$, we have:

$$(z - mid(\mathbf{J})x)_i = \sum_{j=1}^{n_g} \sum_{k=1}^{m} rad(\mathbf{J})_{ik} \overline{M}_{kj} \psi_{ik} \xi_j$$

and, since $|\psi_{ik}\xi_j| \leq 1$, $|(z - mid(\mathbf{J})x)_i| \leq \sum_{j=1}^{n_g} \sum_{k=1}^{m} rad(\mathbf{J})_{ik} |\overline{M}_{kj}| = P_{ii}$ and $(z - mid(\mathbf{J})x) \in P\mathcal{B}_\infty^n$. Therefore, we have $z \in mid(\mathbf{J})X \oplus P\mathcal{B}_\infty^n = \triangleleft(\mathbf{Z})$. □

Lemma 4.3 ([31])**.** *Given* $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n$ *a class* $\mathcal{C}^1$ *function,* $X = h \oplus M\mathcal{B}_\infty(A,b) \subseteq \mathbb{R}^m$ *a constrained zonotope and* $\mathbf{J} \in \mathbb{I}^{n \times m}$ *an interval matrix satisfying* $\nabla^\intercal \varphi(X) \subseteq \mathbf{J}$*, we have*

$$\begin{aligned} \varphi(X) &\subseteq \varphi(h) \oplus \triangleleft(\mathbf{J}(X - h)) \\ &= \varphi(h) \oplus mid(\mathbf{J}) M \mathcal{B}_\infty(A,b) \oplus P\mathcal{B}_\infty^n, \end{aligned} \tag{4.16}$$

where P *is defined as in* (4.15)*.*

Proof. From the application of the mean-value theorem, given $y \in \varphi(x)$, with $x \in X$, there is a $J \in \mathbf{J}$ such that:

$$y = \varphi(h) + J(x - h).$$

Then, from Lemma 4.2, we have $J(x-h) \in \triangleleft(\mathbf{J}(X-h))$ and thus $y = \varphi(h) + J(x-h) \in \varphi(h) \oplus \triangleleft(\mathbf{J}(X-h))$. □

Finally, in Property 4.4 the mean-value constrained zonotope extension is applied to generalize the disturbance propagation methods proposed in Section 4.2 to constrained zonotopes.

Property 4.4. *Consider system* (2.4a)*, a centered constrained zonotope* $\overline{\mathcal{DM}} \subseteq \mathbb{R}^n$ *and an interval matrix* $\mathbf{J}_\pi \in \mathbb{I}^{n \times n}$*, with* $\mathcal{DM} \subseteq \overline{\mathcal{DM}}$ *and* $\nabla_x^\intercal f_\pi(\mathcal{Z}_\pi) \subseteq \mathbf{J}_\pi$*. Consider the sets* $\mathcal{S}_z(j),\ \mathcal{S}_z^\mu(j),\ j = 0 \dots N$ *recursively defined by:*

(i) $\mathcal{S}_z(0)$ *and* $\mathcal{S}_z^\mu(0)$ *are centered constrained zonotopes such that* $\mathcal{W} \subseteq \mathcal{S}_z(0)$*,* $\overline{\mathcal{W}} \subseteq \mathcal{S}_z^\mu(0)$*.*

(ii) $\mathcal{S}_z(j) = \triangleleft(\mathbf{J}_\pi \mathcal{S}_z(j-1))$ *and* $\mathcal{S}_z^\mu(j) = \triangleleft(\mathbf{J}_\pi \mathcal{S}_z^\mu(j-1)) \oplus \overline{\mathcal{DM}}$*,* $\quad j = 1 \dots N$*.*

The sets $\mathcal{S}_z(j)$ *and* $\mathcal{S}_z^\mu(j)$ *satisfy Conditions 2.1 and 3.1, respectively.*

Proof. Analogous to the proofs of Properties 4.1 and 4.2, with the mean-value CZ-extension $\triangleleft(\cdot)$ replacing the zonotopic one. □

Remark 4.4. *From Theorem 4.4, at each propagation step the number of generators and constraints of* $\mathcal{S}_z^{\mu}(j)$ *increase by* $n + n_g^D$ *and* n_c^D*, respectively, where* n_g^D *and* n_c^D *are the number of generators and constraints of* $\overline{\mathcal{DM}}$*. Analogously,* $\mathcal{S}_z(j)$ *has* n *more generators than* $\mathcal{S}_z(j-1)$*. The number of generators and constraints of the disturbance propagation sets can nonetheless be limited via considering the algorithms for reducing the number of generators and constraints of constrained zonotopes proposed in [35].*

Remark 4.5. *Notice, from Lemma 4.2, that an outer zonotopic approximation of a constrained zonotope is needed in order to compute the mean-value extension and thus the disturbance propagation sets. This can be computed via reducing the number of constraints of the constrained zonotope via the method proposed in [35] until there are no more constraints, i.e. we have a simple zonotope.*

Remark 4.6. *The Pontryagin difference of polytopes and constrained zonotopes, necessary in this case for the recursive constraint tightening of Eq.* (2.8)*, cannot be made algebraically as in the zonotopic case (Appendix A) and thus has a higher computational cost. However, it is equivalent to the linear program of maximizing* $h^{\intercal}(c + G\xi)$ *subject to* $\xi \in \mathcal{B}_{\infty}(A, b)$ *[31].*

Recapitulation

In this chapter, the zonotopic disturbance propagation method, based on the mean-value extension [1], was presented and compared to the approach based on Lipschitz constants. In particular, the following topics were discussed:

- Disturbance propagation method based on Lipschitz constants: A method to compute disturbance propagation sets $\mathcal{S}(j)$ using a Lipschitz constant L_x of the model function was presented. These sets are defined by an upper bound on the norm $x_{k+j|k+1} - x_{k+j|k}$, which is multiplied by L_x at each prediction step, thus propagating the worst-case gain in all directions.

- Zonotopic methods: Zonotopic methods to compute sets $\mathcal{S}(j)$ satisfying Conditions 2.1 and 3.1 were proposed, based on the product of a zonotope by an interval matrix and the mean-value extension of zonotopes [1].

- Comparison of criteria: The zonotopic and Lipschitz criteria were compared, and the zonotopic approach was shown to be less conservative than the infinity-norm Lipschitz method. The zonotopes were also shown to be reduced to the optimal disturbance propagation sets $\mathcal{S}^*(j)$ in the linear case.

- Extension to constrained zonotopes: The disturbance propagation methods were extended to consider constrained zonotopes [35], which can reduce conservatism in case the initial disturbance set $\mathcal{W}$ is asymmetric.

Chapter 5

Robust NMPC for Tracking

In this chapter, the robust NMPC algorithm for regulation proposed in Section 2.4 will be extended to the reference tracking case. For such, the references, considered piece-wise constant, must be associated with admissible equilibrium points, similar to the steady-state correction presented in Section 3.1.

The proposed robust nonlinear predictive controller for tracking is then presented, as well as the associated assumptions to be satisfied by the cost functions and terminal ingredients in order to assure recursive feasibility and Input-to-State Stability (ISS). An artificial reference is included as an additional variable in the optimization problem, such that the controller feasibility is independent of the desired steady-state. Finally, simplified methods for the computation of the terminal ingredients are presented.

5.1 Equilibrium Condition

A nominal equilibrium point of system (2.1), associated to the steady-state output y_s, can be represented by the pair (x_s, u_s), with the equilibrium condition given by

$$\begin{aligned} x_s &= f(x_s, u_s) \\ y_s &= h(x_s, u_s). \end{aligned} \tag{5.1}$$

Given a set of constraints $(x_k, u_k) \in \mathcal{Z}$, the set of admissible equilibrium points $\mathcal{Z}_s(\mathcal{Z})$ and the set of reachable references $\mathcal{Y}_s(\mathcal{Z})$ of the system are thus given by

$$\mathcal{Z}_s(\mathcal{Z}) = \{(x, u) \in \hat{\mathcal{Z}} \colon x = f(x, u)\}, \tag{5.2}$$

$$\mathcal{Y}_s(\mathcal{Z}) = \{y = h(x, u) \colon (x, u) \in \mathcal{Z}_s(\mathcal{Z})\}, \tag{5.3}$$

where $\hat{\mathcal{Z}} = \{z \in \mathbb{R}^{n+m} \colon z + e \in \mathcal{Z},\ \forall \|e\| < \epsilon\}$, with $\epsilon > 0$ an auxiliary parameter included in order to avoid the boundaries of the constraint set. Notice that equilibrium points arbitrarily close to the boundaries can be considered by reducing the value of ϵ[1].

Assumption 5.1. *For a given reachable target $y_s \in \mathcal{Y}_s(\mathcal{Z})$, consider that there exists a unique associated steady-state $x_s = \varrho_x(y_s)$, $u_s = \varrho_u(y_s)$, where $\varrho_x \colon \mathcal{Y}_s(\mathcal{Z}) \to \mathbb{R}^n$ and $\varrho_u \colon \mathcal{Y}_s(\mathcal{Z}) \to \mathbb{R}^m$ are Lipschitz continuous functions, such that:*

$$\begin{aligned} x_s &= f(x_s, u_s), \\ y_s &= h(x_s, u_s). \end{aligned} \tag{5.4}$$

Remark 5.1. *Notice that Assumption 5.1 is analogous to Assumption 3.1, with ϱ_x, ϱ_u assuming a similar role to g_x, g_v. Therefore, through the implicit function theorem [18], Assumption 5.1 is satisfied if $m = p$ and the following matrix is nonsingular*

$$\begin{pmatrix} A(x_s, u_s) - I_n & B(x_s, u_s) \\ C(x_s, u_s) & D(x_s, u_s) \end{pmatrix},$$

where A, B, C and D represent the linearized system (2.1) *at (x_s, u_s), for all admissible equilibrium points $(x_s, u_s) \in \mathcal{Z}_s(\mathcal{Z})$.*

5.2 Robust NMPC for Tracking

In this section, the NMPC algorithm for tracking is presented. The proposed strategy also applies nominal predictions and tightened constraints. However, aiming to increase the domain of attraction and avoid feasibility loss due to reference changes, an artificial reference y_s is considered as an additional variable in the optimization problem [22]. Convergence to the actual reference y_r is then ensured via the insertion of the term $V_O(y_s - y_t)$, which penalizes the difference between real and artificial references, to the cost function.

Due to the presence of the artificial reference, it is necessary to extend the idea of Robust Positive Invariant sets presented in Chapter 2 to the tracking case. where the equilibrium point is any (x_s, u_s) with $y_s = h(x_s, u_s)$, rather than just the origin.

Definition 5.1 (Robust Positively Invariant set for tracking)**.** *Consider a set $\Gamma \subseteq \mathbb{R}^{n+p}$ and a control law $u_k = \kappa_t(x_k, y_s)$. Γ is a Robust Positively Invariant set for tracking for system* (2.1) *subject to disturbances $w_k \in \mathcal{W}_t$ if for all $(x, y_s) \in \Gamma$ and $w \in \mathcal{W}_t$, then $(f(x, \kappa_t(x, y_s)) + w, y_s) \in \Gamma$.*

Definition 5.1 means that once state and reference are inside the set Γ, the control law $\kappa_t \colon \mathbb{R}^{n+p} \to \mathbb{R}^m$, with a fixed artificial reference, guarantees that the next state is also in the set, regardless of the disturbance $w_k \in \mathcal{W}_t$ (robust positively invariant property).

[1] In practice, due to numerical restrictions of the optimization solver, in practical implementations ϵ cannot be arbitrarily small.

Based on the current state x_k and the desired setpoint y_t, the NMPC solves the optimal control problem $P_N^t(x_k, y_t)$ defined as:

$$\min_{\hat{\mathbf{v}}, y_s} \sum_{j=0}^{N-1} L(x_{k+j|k} - x_s, \pi(v_{k+j|k}, x_{k+j|k}) - u_s) \ + \ V_f(x_{k+N|k} - x_s, y_s) \ + V_O(y_s - y_t) \tag{5.5a}$$

$s.t:$

$$x_{k+j+1|k} = f_\pi(x_{k+j|k}, v_{k+j|k}), \qquad j \in \mathbb{Z}_{[0,N-1]}, \tag{5.5b}$$

$$(x_{k+j|k}, v_{k+j|k}) \in \mathcal{Z}_\pi(j), \qquad j \in \mathbb{Z}_{[0,N-1]}, \tag{5.5c}$$

$$x_s = \varrho_x(y_s), \ u_s = \varrho_u(y_s), \tag{5.5d}$$

$$(x_{k+N|k}, y_s) \in \Gamma, \tag{5.5e}$$

where $\hat{\mathbf{v}} = (v_{k|k}, v_{k+1|k}, \ldots, v_{k+N-1|k})$ and y_s are respectively the virtual inputs and artificial reference, L, V_f and V_O are respectively the stage, terminal and offset costs, $\mathcal{Z}_\pi(j)$ are the tightened constraints, recursively computed via Eq. (2.8) from disturbance propagation sets $\mathcal{S}(j)$ which satisfy Condition 2.1, and Γ is the terminal set.

Notice that, due to the freedom provided by the artificial reference, the feasibility of $P_N^t(x_k, y_t)$ and, therefore, the domain of attraction $\mathcal{X}_N$, is independent of y_t. For a given $x_k \in \mathcal{X}_N$ and $y_t \in \mathbb{R}^p$, $\hat{\mathbf{v}}^*(x_k, y_t)$ and $y_s^*(x_k, y_t)$ are respectively the virtual input sequence and artificial reference that solve the MPC optimization problem $P_N^t(x_k, y_t)$, with $V_N^*(x_k, y_t)$ the associated minimal cost. Based on the receding horizon policy, the proposed NMPC control law is defined as follows:

$$u_k = \kappa_r(x_k, y_t) = \pi(x_k, v_k^*), \tag{5.6}$$

where v_k^* is obtained from the solution of $P_N^t(x_k, y_t)$ at each sampling instant. Notice that (5.5d) can alternatively be replaced by:

$$\begin{aligned} x_s &= f(x_s, u_s), \\ y_s &= h(x_s, u_s), \end{aligned} \tag{5.7}$$

being x_s and u_s additional optimization variables, thus eliminating the need for an explicit knowledge of the functions $\varrho_x(\cdot)$ and $\varrho_u(\cdot)$ [22].

The stage cost $L\colon \mathbb{R}^{n+m} \to \mathbb{R}$, the offset cost $V_O\colon \mathbb{R}^p \to \mathbb{R}$ and the set of feasible equilibria $\mathcal{Y}_t$ must satisfy the following assumptions:

Assumption 5.2.

(i) The stage cost function is positive definite and uniformly continuous, such that:

$$L(x, u) \geq \alpha_L(\|x\|), \tag{5.8a}$$

$$|L(x_1, u_1) - L(x_2, u_2)| \leq \lambda_x(\|x_1 - x_2\|) + \lambda_u(\|u_1 - u_2\|), \tag{5.8b}$$

where λ_x and λ_u are $\mathcal{K}$-functions and α_L is a $\mathcal{K}_\infty$-function.

(ii) The set of feasible references $\mathcal{Y}_t = \{y_s \in \mathbb{R}^p \colon (\varrho_x(y_s), y_s) \in \Gamma\}$ is a convex, compact, non-empty subset of $\mathcal{Y}_s(\mathcal{A}_N)$, where $\mathcal{A}_N \subseteq \mathcal{Z}(N) = \{(x, \pi(x,v)) \in \mathbb{R}^{n+m} \colon (x,v) \in \mathcal{Z}_\pi(N)\}$ is the N-step-ahead admissible set.

(iii) The offset cost function is positive definite, uniformly continuous and strictly convex, thus assuring that the minimizer

$$y_s^o = arg \min_{y_s \in \mathcal{Y}_t} V_O(y_s - y_t) \tag{5.9}$$

is unique. Furthermore, for any $y_t \in \mathbb{R}^p$ and $y_s \in \mathcal{Y}_t$, we have

$$V_O(y_s - y_t) - V_O(y_s^o - y_t) \geq \alpha_O(\|y_s - y_s^o\|), \tag{5.10}$$

where α_O is a $\mathcal{K}_\infty$-function.

Remark 5.2. *In particular, quadratic stage and offset costs can be considered. For the stage cost, $L(x,u) = x^\intercal Q x + u^\intercal R u$ is positive definite and uniformly continuous in $\mathcal{Z}$ for any $Q \succ 0 \in \mathbb{R}^{n\times n}, R \succeq 0 \in \mathbb{R}^{m\times m}$. For the offset cost, $V_O(y) = y^\intercal T y$ is positive definite, strictly convex and uniformly continuous in $\mathcal{Y}_t$ for any $T \succ 0 \in \mathbb{R}^{p\times p}$. Furthermore,* (5.10) *is satisfied, as shown in Lemma B.3, Appendix B.*

The terminal control law $v_t \colon \mathbb{R}^{n+p} \to \mathbb{R}^m$, terminal cost $V_f \colon \mathbb{R}^{n+p} \to \mathbb{R}$ and terminal set $\Gamma \subseteq \mathbb{R}^{n+p}$, where $u_t(x, y_s) \coloneqq \pi(x, v_t(x, y_s))$, must satisfy the following assumptions:

Assumption 5.3.

(i) The terminal control law must satisfy $u_t(x_s, y_s) = u_s$ for all admissible equilibrium points $(x_s, u_s) = (\varrho_x(y_s), \varrho_u(y_s))$, $y_s \in \mathcal{Y}_t$.

(ii) The terminal set $\Gamma \subseteq \Lambda_N$ where $\Lambda_N = \{(x,y) \in \mathbb{R}^n \times \mathcal{Y}_t \colon (x, u_t(x,y)) \in \mathcal{A}_N\}$ is an admissible robust positively invariant set for tracking subject to $u = u_t(x, y_s)$ for any $w \in \mathcal{S}(N)$. That is, $(x, y_s) \in \Gamma \subseteq \Lambda_N \Rightarrow (f(x, u_t(x, y_s)), y_s) \oplus (\mathcal{S}(N) \times \{0\}) \subseteq \Gamma$.

(iii) The terminal cost function $V_f(x - x_s, y_s)$ must be an uniformly continuous Lyapunov function for the system $x_{k+1} = f(x_k, u_t(x_k, y_s))$, with constants $b > 0, a > 1 \in \mathbb{R}$ such that for all $(x, y_s) \in \Gamma$ we have:

$$0 \leq V_f(x - x_s, y_s) \leq b\|x - x_s\|^a, \tag{5.11a}$$

$$|V_f(x_1, y_s) - V_f(x_2, y_s)| \leq \delta(\|x_1 - x_2\|), \tag{5.11b}$$

$$V_f(f(x, u_t(x, y_s)) - x_s, y_s) - V_f(x - x_s, y_s) \leq -L(x - x_s, u_t(x, y_s) - u_s), \tag{5.11c}$$

where $x_s = \varrho_x(y_s)$, $u_s = \varrho_u(y_s)$ and δ is a $\mathcal{K}$-function.

Remark 5.3. *In particular, a quadratic terminal cost* $V_f(x, y_s) = x^\intercal P x,\ \ P \succ 0$ *can be considered, since it is uniformly continuous in* $\mathcal{Z}$ *and* $x^\intercal P x \leq \lambda_{M,P} \|x\|_2^2$*, where* $\lambda_{M,P}$ *is the largest eigenvalue of* P*, thus Eq.* (5.11a) *is satisfied with* $b = \lambda_{M,P}$ *and* $a = 2$*. Similarly to the regulation case, also considering a linear terminal control law and quadratic stage cost, Eq.* (5.11c) *can then be converted to a Linear Matrix Inequality (LMI) problem, as detailed in Section 5.3.*

Notice that these assumptions are similar to the ones presented in Chapter 2 and related works [34, 22], but extended to the NMPC problem with bounded disturbances and piecewise constant references. Finally, the following mild additional assumptions are included in order to ensure Input-to-State stability:

Assumption 5.4.

(i) There exist positive constants $s_0, c_s > 0 \in \mathbb{R}$ *such that*

$$\alpha_O(s) \geq c_s s^a, \quad \forall s \in \mathbb{R}^+,\ s < s_0. \tag{5.12}$$

(ii) The origin is an interior point of $\mathcal{S}(N)$*, i.e. there exists* $\varepsilon_s > 0 \in \mathbb{R}$ *such that* $\|x\| < \varepsilon_s \Rightarrow x \in \mathcal{S}(N)$.

Remark 5.4. *Notice that since the constants* $s_0, c_s > 0$ *can be arbitrarily small, Assumption 5.4(i) is equivalent to* $lim_{s\to 0} \frac{\alpha_O(s)}{s^a} > 0$*. In particular, if quadratic terminal and offset costs are used,* $a = 2$ *and, as shown in Lemma B.3, Appendix B,* α_O *can be defined as* $\alpha_O(s) = \lambda_{m,T} s^2$*, where* $\lambda_{m,T}$ *is the smallest eigenvalue of* $T \succ 0$*. Therefore, Eq.* (5.12) *can be trivially satisfied by making* $c_s = \lambda_{m,T}$.

Remark 5.5. *Since the origin is an interior point of* $\mathcal{W}$*, Assumption 5.4(ii) is a consequence of Condition 2.1 and the implicit function theorem if* $\nabla_x^\intercal f_\pi(x, v)$ *is nonsingular for a given* $(x, v) \in \mathcal{Z}_\pi(N)$*. Nonetheless, if the origin is not an interior point of* $\mathcal{S}(N)$*, which in the zonotopic case is equivalent to* $\mathcal{S}_z(N)$ *being degenerated* [2]*, it suffices to consider a modified disturbance propagation set* $\overline{\mathcal{S}}(N) = \mathcal{S}(N) \oplus \varepsilon_s \mathcal{B}_\infty^n$.

The main properties of this NMPC algorithm for tracking are presented in Lemma 5.1, and Theorem 5.1. They ensure recursive feasibility and Input-to-State Stability (ISS) of system (2.1) subject to the NMPC control law (5.6).

Lemma 5.1 (Recursive Feasibility)**.** *Let* $\mathcal{X}_N$ *be the domain of attraction of the NMPC controller* (5.6)*. Then, the following properties hold:*

[2] A zonotope $Z = c \oplus G\mathcal{B}_\infty^{n_g} \subseteq \mathbb{R}^n$ is said to be degenerated if its generator matrix $G \in \mathbb{R}^{n \times n_g}$ is not full line rank.

(i) *For any* $y_t \in \mathbb{R}^p$ *and* $x_k \in \mathcal{X}_N$*, the NMPC control law provided by Eq.* (5.6)*, namely* $u_k = \kappa_r(x_k, y_t)$*, is such that* $x_{k+1} = f(x_k, u_k) + w_k \in \mathcal{X}_N,\ \forall w_k \in \mathcal{W}$.

(ii) *Given* $\hat{\mathbf{v}}^*(x_k, y_t) = (v_0^*, v_1^*, \ldots, v_{N-1}^*)$ *and* $y_s^*(x_k, y_t) = y_s^*$*, then the virtual control sequence* $\hat{\mathbf{v}}^c = (v_1^*, \ldots, v_{N-1}^*, v_t(x_{k+N|k}, y_s^*))$ *and artificial reference* $y_s^c = y_s^*$ *define a feasible (candidate) solution of* $P_N^t(x_{k+1}, \overline{y}_t)$*, for any* $\overline{y}_t \in \mathbb{R}^p$ *and* $w_k \in \mathcal{W}$.

Proof. Consider the optimal solution at k, $\hat{\mathbf{v}}^*_{[k,k+N-1]}$, $y^*_{s,k}$, and the candidate solution at $k+1$, $\hat{\mathbf{v}}^c = (v^*_{k+1|k}, \ldots, v^*_{k+N-1|k}, v_t(x^*_{k+N|k}))$, $y^c_{s,k+1} = y^*_{s,k}$, which provide the optimal and candidate predictions given by $x^*_{k+j|k} = f_\pi(x^*_{k+j-1|k}, v^*_{k+j-1|k})$ and $x^c_{k+j|k+1} = f_\pi(x^c_{k+j-1|k+1}, v^c_{k+j-1|k+1})$, respectively. Now, given $x_{k+1} = f_\pi(x_k, v_k^*) + w_k$, with $w_k \in \mathcal{W}$, then from Condition 2.1, $x^c_{k+j|k+1} \in x^*_{k+j|k} \oplus \mathcal{S}(j-1),\ j = 1 \ldots N$ is ensured.

Moreover, due to the definition of the NMPC problem $P_N(x_k, y_t)$, $(x^*_{k+j|k}, v^*_{k+j|k}) \in \mathcal{Z}_\pi(j)$, $\forall j = 1 \ldots N-1$ and, from $(x^*_{k+N|k}, y_s^*) \in \Gamma \subseteq \Lambda_N$, $(x^*_{k+N|k}, v_t(x^*_{k+N|k}, y_s^*)) \in \mathcal{Z}_\pi(N)$. Hence, the following candidate condition holds for $j = 1 \ldots N$:

$$(x^c_{k+1+j|k+1}, v^c_{k+1+j|k+1}) \in (x^*_{k+1+j|k}, v^*_{k+1+j|k}) \oplus (\mathcal{S}(j) \times \{0\}) \subseteq \mathcal{Z}_\pi(j),$$

where for simplicity of notation $v_N^* = v_t(x^*_{k+N|k}, y_s^*)$ was defined. For the terminal constraint, we use the fact that Γ is a robust positively invariant set for tracking. Therefore, $(x^*_{k+N|k}, y_s^*) \in \Gamma$ guarantees that

$$(f_\pi(x^*_{k+N|k}, v_t(x^*_{k+N|k}, y_s^*)) + w, y_s^*) \in \Gamma, \quad \forall w \in \mathcal{S}(N).$$

Since $v^c_{k+N|k+1} = v_t(x^*_{k+N|k}, y_s^*)$, $x^c_{k+1+N|k+1} \in f_\pi(x^*_{k+N|k}, v_t(x^*_{k+N|k}, y_s^*)) \oplus \mathcal{S}(N)$ and thus $(x^c_{k+1+N|k+1}, y_s^*) \in \Gamma$. Therefore, $\hat{\mathbf{v}} = \hat{\mathbf{v}}^c$, $y_s = y_s^c$ define a feasible solution for $P_N^t(x_{k+1}, \overline{y}_t)$ and $x_{k+1} \in \mathcal{X}_N$. □

Remark 5.6. *Notice that, given an admissible artificial reference* $y_s \in \mathcal{Y}_t$*, with* $v_s = v_t(x_s, y_s)$ *and* $x_s = \varrho_x(y_s)$*, then the virtual control sequence* $\hat{\mathbf{v}} = (v_s, \ldots, v_s)$ *is a feasible candidate for the problem* $P_N^t(x_s, y_t)$*. Therefore, the set of admissible equilibrium states* $\mathcal{X}_t = \{x \in \mathbb{R}^n \colon x = \varrho_x(y_s),\ y_s \in \mathcal{Y}_t\}$ *is a subset of* $\mathcal{X}_N$*. Moreover, feasibility is not lost due to setpoint changes, because (i)* y_s *is a free decision variable, and (ii) the constraints of the optimization problem do not depend on* y_t.

Theorem 5.1 (Input-to-State Stability)**.** *Assume that Assumptions 5.1, 5.2, 5.3 and 5.4 hold. The system* (2.1) *subject to the NMPC control law* (5.6) *is Input-to-State Stable in* $\mathcal{X}_N$*. That is, for any* $x_0 \in \mathcal{X}_N$ *and constant setpoint* $y_t \in \mathbb{R}^p$*, with* $u_k = \kappa_r(x_k, y_t)$*,* $w_k \in \mathcal{W}$*,* $\forall k \in \mathbb{N}$*, the following inequality holds:*

$$\|x_k - x_s^o\| \le \beta\left(\|x_0 - x_s^o\|, k\right) + \gamma(\left\|\mathbf{w}_{[0,k]}\right\|), \tag{5.13}$$

where y_s^o *is given as in* (5.9)*,* $x_s^o = \varrho_x(y_s^o)$*, and* $\beta(\cdot)$ *and* $\gamma(\cdot)$ *are respectively a* $\mathcal{KL}$*-function and a* $\mathcal{K}$*-function.*

Proof. Define $W^*(x_k, y_t) = V_N^*(x_k, y_t) - V_O(y_s^o - y_t)$, where $V_O(y_s^o - y_t)$ is the optimal constant value of the offset cost. The feasible candidate $\hat{\mathbf{v}}^c$ will be used to show that $W^*(x_k, y_t)$ is an ISS-Lyapunov function for system (2.1) subject to the control law (5.6). Being L_g a Lipschitz constant for ϱ_x in $\mathcal{Y}_t$, i.e. $\|\varrho_x(y_b) - \varrho_x(y_a)\| \leq L_g \|y_b - y_a\|$, $\forall y_a, y_b \in \mathcal{Y}_t$, then:

$$\begin{aligned}
W^*(x_k, y_t) \geq & L(x_k - x_{s,k}^*, \pi(x_k, v_k^*) - u_{s,k}^*) + V_O(y_{s,k}^* - y_t) - V_O(y_s^o - y_t) \\
\geq & \alpha_L(\|x_k - x_{s,k}^*\|) + \alpha_O(\|y_{s,k}^* - y_s^o\|) \\
\geq & \alpha_L(\|x_k - x_{s,k}^*\|) + \alpha_O(L_g^{-1}\|x_{s,k}^* - x_s^o\|)
\end{aligned}$$

Now, defining the $\mathcal{K}_\infty$-function $\alpha_W(s) = \min\{\alpha_L(s/2), \alpha_O(L_g^{-1}s/2)\}$ and noticing that via the triangular inequality $\|x_k - x_s^o\| \leq \|x_k - x_{s,k}^*\| + \|x_{s,k}^* - x_s^o\| \leq 2\max\{\|x_k - x_{s,k}^*\|, \|x_{s,k}^* - x_s^o\|\}$, we have

$$\begin{aligned}
W^*(x_k, y_t) \geq & \alpha_W(2\|x_k - x_{s,k}^*\|) + \alpha_W(2\|x_{s,k}^* - x_s^o\|) \\
\geq & \max\{\alpha_W(2\|x_k - x_{s,k}^*\|), \alpha_W(2\|x_{s,k}^* - x_s^o\|)\} \\
= & \alpha_W(\max\{2\|x_k - x_{s,k}^*\|, 2\|x_{s,k}^* - x_s^o\|\}) \\
\geq & \alpha_W(\|x_k - x_s^o\|).
\end{aligned}$$

From Lemma B.4, $W^*(x_k, y_t) \leq V_f(x_k - x_s^o, y_s^o) \leq b\|x_k - x_s^o\|^a$ if $\|x_k - x_s^o\| \leq \varepsilon_s$. Moreover, $W^*(x_k, y_t)$ is bounded in $\mathcal{X}_N$ from the continuity of the cost functions and compactness of the constraints, and thus, from Lemma B.2, there exists $\bar{b} > 0 \in \mathbb{R}$ such that

$$\alpha_W(\|x_k - x_s^o\|) \leq W^*(x_k, y_t) \leq \bar{b}\|x_k - x_s^o\|^a, \quad \forall x_k \in \mathcal{X}_N. \tag{5.14}$$

Now, the feasible candidate is used to define $W^c(x_{k+1}, y_t) = V_N^c(x_{k+1}, y_t) - V_O(y_s^o - y_t)$, with $\hat{\mathbf{v}}^c = (v_1^*, \ldots, v_{N-1}^*, v_t(x_{k+N|k}^*, y_{s,k}^*))$ and $y_{s,k+1}^c = y_{s,k}^*$, as previously discussed. For simplicity of notation, consider $u_{k+j|k}^* = \pi(x_{k+j|k}^*, v_{k+j|k}^*)$, $j = 0 \ldots N-1$, $u_{k+1+j|k+1}^c = \pi(x_{k+1+j|k+1}^c, v_{k+1+j|k+1}^c)$, $j = 0 \ldots N-1$, $u_{k+N|k}^* = u_t(x_{k+N|k}^*, y_{s,k}^*)$ and $x_{k+N+1|k}^* = f(x_{k+N|k}^*, u_{k+N|k}^*)$. From the uniform continuity of the model, there exists a $\mathcal{K}$-function $\sigma_x(\cdot)$ such that $\left\|x_{k+j|k+1}^c - x_{k+j|k}^*\right\| \leq \sigma_x^{j-1}(\|w_k\|)$, $j = 1 \ldots N+1$. Therefore, omitting the dependence of the terminal cost on y_s for presentation simplicity, we have:

$$\begin{aligned}
|L(x_{k+j|k+1}^c - x_{s,k+1}^c, u_{k+j|k+1}^c - u_{s,k+1}^c) - L(x_{k+j|k}^* - x_{s,k}^*, u_{k+j|k}^* - u_{s,k}^*)| \leq \lambda_x(\sigma_x^{j-1}(\|w_k\|)), \\
|V_f(x_{k+N+1|k+1}^c - x_{s,k+1}^c) - V_f(x_{k+N+1|k}^* - x_{s,k}^*)| \leq \delta(\sigma_x^N(\|w_k\|)),
\end{aligned}$$

for $j = 1 \ldots N$. An upper bound for the candidate objective function can then be written:

$$\begin{aligned}
V_N^c(x_{k+1}, y_t) \leq & \sum_{j=1}^{N-1} L(x_{k+j|k}^* - x_{s,k}^*, u_{k+j|k}^* - u_{s,k}^*) + L(x_{k+N|k}^* - x_{s,k}^*, u_{k+N|k}^* - u_{s,k}^*) \\
& + V_f(x_{k+N+1|k}^* - x_{s,k}^*) + \sum_{j=0}^{N-1} \lambda_x(\sigma_x^j(\|w_k\|)) + \lambda_V(\sigma_x^N(\|w_k\|)).
\end{aligned}$$

Now define $\Delta W = W^c(x_{k+1}, y_t) - W^*(x_k, y_t) = V_N^c(x_{k+1}, y_t) - V_N^*(x_k, y_t)$. Hence, from Assumption 5.3 (iii) and Assumption 5.2 (i), the following bound for ΔWcan be established:

$$\begin{aligned}
\Delta W \leq & - L(x_k - x_{s,k}^*, u_k - u_{s,k}^*) + V_f(x_{k+N+1|k}^* - x_{s,k}^*) - V_f(x_{k+N|k}^* - x_{s,k}^*) \\
& \quad + L(x_{k+N|k}^* - x_{s,k}^*, u_{k+N|k}^* - u_{s,k}^*) + \sum_{j=0}^{N-1} \lambda_x(\sigma_x^j(\|w_k\|)) + \lambda_V(\sigma_x^N(\|w_k\|)) \\
\leq & - L(x_k - x_{s,k}^*, u_k - u_{s,k}^*) + \theta(\|w_k\|) \\
\leq & - \alpha_L(\|x_k - x_{s,k}^*\|) + \theta(\|w_k\|),
\end{aligned}$$

where $\theta(\|w_k\|) = \sum_{j=0}^{N-1} \lambda_x(\sigma_x^j(\|w_k\|)) + \lambda_V(\sigma_x^N(\|w_k\|))$ is a $\mathcal{K}$-function. Then, due to the optimality of the effective solution, the following cost difference is verified:

$$W^*(x_{k+1}, y_t) - W^*(x_k, y_t) \leq -\alpha_L(\|x_k - x_{s,k}^*\|) + \theta(\|w_k\|). \tag{5.15}$$

From Lemma B.5, defining $\overline{\alpha}_L = \alpha_L \circ \alpha_V^{-1}$ a $\mathcal{K}_\infty$-function, $\overline{\alpha}_L(\|x_k - x_s^o\|) \leq \alpha_L(\left\|x_k - x_{s,k}^*\right\|)$. Therefore, from Eq. (5.15), we have

$$W^*(x_{k+1}, y_t) - W^*(x_k, y_t) \leq -\overline{\alpha}_L(\|x_k - x_s^o\|) + \theta(\|w_k\|). \tag{5.16}$$

Finally, from the inequalities (5.14) and (5.16), $W^*(x_k, y_t)$ is an ISS-Lyapunov function for the NMPC control system and, through Lemma B.1, Eq. (5.13) is satisfied. □

5.3 Simplified Terminal Ingredients

In this section, the choice of terminal control law, cost and constraints satisfying the assumptions presented in Section 5.2 is considered. A method to obtain a linear terminal control law and associated quadratic terminal cost is presented. Then, the problem of computing a polyhedral terminal set based on linear models is considered.

Consider for simplicity a quadratic stage cost $L(x, u) = x^\intercal Q x + u^\intercal R u$, as presented in Section 2.5, and let $V_f(x, y_s) = x^\intercal P x$ and $(u - u_s) = K_t(x - x_s)$ be the quadratic terminal cost and linear terminal control law, respectively. The terminal control law can be rewritten as $u = u_t(x, y_s) = K_t x + (u_s - K_t x_s) = K_t x + \theta$, where $\theta(y_s) = u_s - K_t x_s$ represents the *offset* due to the artificial reference. Notice that the conditions: $u_t(x_s, y_s) = u_s$, uniform continuity of V_f in $\mathcal{Z}$, and $V_f(x) \leq \lambda_{P.M} \|x\|_2^2$, where $\lambda_{P,M} > 0$ is the biggest eigenvalue of P, are directly ensured for any $K_t \in \mathbb{R}^{m\times n}$ and $P \succ 0 \in \mathbb{R}^{n\times n}$.

Therefore, the choice of the pair (K_t, P), as in the regulation case, is based on stabilizing the system $x_{k+1} = f(x_k, K_t x_k + \theta)$, satisfying the decreasing cost assumption (5.11c) for any admissible $(x, y_s) \in \Lambda_N$. Theorem 5.2 shows that the same Linear Matrix Inequality presented in Theorem 2.2 for regulation can be applied to the tracking case.

Theorem 5.2. *Consider the nonlinear system* (2.1)*, the terminal control law* $u_t(x, y_s) = K_t x + \theta$ *and terminal cost* $V_t(x) = x^\intercal P x$*,* $P \succ 0$*, and let* $\mathbf{A} \in \mathbb{I}^{n\times n}$*,* $\mathbf{B} \in \mathbb{I}^{n\times m}$ *be interval matrices satisfying* $\nabla_x^\intercal f(\mathcal{A}_N) \in \mathbf{A}$*,* $\nabla_u^\intercal f(\mathcal{A}_N) \in \mathbf{B}$*.*

If, for any $A_\ell \in \mathbf{A}$ *and* $B_\ell \in \mathbf{B}$*, we have*

$$(A_\ell + B_\ell K_t)^\intercal P (A_\ell + B_\ell K_t) - P + (Q + K_t^\intercal R K_t) \preceq 0, \tag{5.17}$$

then the decreasing cost assumption (5.11c) *is satisfied for any* $(x, y_s) \in \Lambda_N$*.*

Proof. This proof follows a similar argument to that of Theorem 2.2. Consider $(x_k, y_s) \in \Lambda_N$, $x_s = \varrho_x(y_s)$ and $\varrho_s = g_u(y_s)$. Defining $\delta x_k = x_k - x_s$ and $\delta x_{k+1} = f(x_k, u_t(x_k, y_s)) - x_s$, noting that $u_t(x_k, y_s) - u_s = K_t \delta x_k$, Eq. (5.11c) can be rewritten as

$$\delta x_{k+1}^\intercal P \delta x_{k+1} - \delta x_k^\intercal P \delta x_k \leq -\delta x_k^\intercal (Q + K_t^\intercal R K_t) \delta x_k. \tag{5.18}$$

Additionally, through the mean-value theorem, $u_t(x_s, y_s) = u_s$ and, from the definition of Λ_N, $(x_s, u_s), (x_k, u_t(x_k, y_s)) \in \mathcal{A}_N$, there exist $A_\ell \in \mathbf{A}, B_\ell \in \mathbf{B}$ such that

$$\begin{aligned} f(x_k, u_t(x_k, y_s)) &= f(x_s, u_s) + A_\ell (x_k - x_s) + B_\ell (u_t(x_k, y_s) - u_s) \\ &= x_s + A_\ell \delta x_k + B_\ell K_t \delta x_k. \end{aligned}$$

Therefore, $\delta x_{k+1} = (A_\ell + B_\ell K_t)\delta x_k$ and thus Eq. (5.11c) is equivalent to

$$\begin{aligned} \delta x_{k+1}^\intercal P \delta x_{k+1}^\intercal - \delta x_k^\intercal P \delta x_k + \delta x_k^\intercal (Q + K_t^\intercal R K_t) \delta x_k &\leq 0, \\ \delta x_k^\intercal ((A_\ell + B_\ell K_t)^\intercal P (A_\ell + B_\ell K_t)) \delta x_k - \delta x_k^\intercal P \delta x_k + \delta x_k^\intercal (Q + K_t^\intercal R K_t) \delta x_k &\leq 0, \\ \delta x_k^\intercal ((A_\ell + B_\ell K_t)^\intercal P (A_\ell + B_\ell K_t) - P + (Q + K_t^\intercal R K_t)) \delta x_k &\leq 0, \end{aligned}$$

which, based on the matrix inequality (5.17), is satisfied. □

Therefore, as discussed in Section 5.2 the pair of matrices (K_t, P) can be computed from the vertices of $\mathbf{A}$ and $\mathbf{B}$ via LTV control methods.

The method for computing the terminal set is based on the iterative algorithm for obtaining polyhedral RPI sets presented in Section 2.6 and exploits the partition method proposed in [37].

First, a pair of matrices (K_t, P) satisfying the decreasing cost assumption (5.11c) is obtained through Theorem 5.2. Then, the following augmented autonomous system is considered

$$\xi_{k+1} = \begin{pmatrix} x_{k+1} \\ \theta_{k+1} \end{pmatrix} = \begin{pmatrix} f(x_k, K_t x_k + \theta_k) \\ \theta_k \end{pmatrix} = f_a(\xi_k), \tag{5.19}$$

where $\xi = (x, \theta) \in \mathbb{R}^{n+m}$ is an augmented vector that represents the original state description and the auxiliary variable $\theta(y_s) = u_s - K_t x_s$. A desired convex set of feasible equilibria

$\hat{\mathcal{Y}}_t \subseteq \mathcal{Y}_s(\mathcal{A}_N)$ is chosen and partitioned into a collection of disjoint sets $\{\mathcal{Y}_j,\ j = 1 \dots N_p\}$ such that $\hat{\mathcal{Y}}_t = \bigcup_j \mathcal{Y}_j$.

For each $j = 1 \dots N_p$, consider a polyhedral compact set $\Psi_j \subseteq \hat{\Psi} = \{(x, \theta) \in \mathbb{R}^{n+m} \colon (x, K_t x + \theta) \in \mathcal{A}_N\}$ such that $(\varrho_x(y_s), \theta(y_s)) \in \Psi_j,\ \forall y_s \in \mathcal{Y}_j$[3]. System (5.19) is then linearized around an equilibrium point ${}^j\xi = (\varrho_x(y_j), \theta(y_j))$, defined by some $y_j \in \mathcal{Y}_j$, resulting in the linear system

$$\delta\xi_{k+1} = A_j\ \delta\xi_k, \tag{5.20}$$

where $\delta\xi_k = \xi_k - {}^j\xi$. An extended disturbance set $\mathcal{W}^j_{amp} = (\mathcal{S}(N) \times \{0\}) \oplus \mathcal{W}^j_{nl}$ is then considered, where $\mathcal{W}^j_{nl}$ bounds the deviation between nonlinear and linearized models, i.e.

$$\delta_j(\xi) = f_a(\xi) - ({}^j\xi + A_j\delta\xi) \in \mathcal{W}^j_{nl}, \quad \forall \xi \in \Psi_j. \tag{5.21}$$

An admissible Robust Positively Invariant (RPI) set $\Phi_j \subseteq \Psi_j$ for the linearized system (5.20), subject to disturbances $w_k \in \mathcal{W}^j_{amp}$, is then obtained through the algorithm presented in Section 2.6. By the definition of $\mathcal{W}^j_{nl}$, Φ_j is also a RPI set for the augmented nonlinear system (5.19) subject to $w_k \in \mathcal{S}(N) \times \{0\}$. Therefore, for any $(x, y_s) \in \Gamma_j = \{(x, y_s) \in \mathbb{R}^n \times \mathcal{Y}_j \colon (x, \theta(y_s)) \in \Phi_j\}$, we have $(f(x, K_t x + \theta(y_s)) + w, y_s) \in \Gamma_j$ for any $w \in \mathcal{S}(N)$, and Γ_j is a Robust Positively Invariant Set for Tracking (TRPI set) for system (2.1) with the terminal control law $u_t(x, y_s) = K_t(x - x_s) + u_s$.

Finally, a convex subset $\mathcal{Y}_t \subseteq \{y_s \in \hat{\mathcal{Y}}_t \colon (\varrho_x(y_s), y_s) \in \bigcup_j \Gamma_j\}$ is considered and the terminal set is defined by $\Gamma = \{(x, y_s) \in \mathbb{R}^n \times \mathcal{Y}_t \colon (x, y_s) \in \bigcup_j \Gamma_j\}$. The convexity of $\mathcal{Y}_t$ is necessary in order to allow for transitions between the TRPI sets Γ_j without feasibility loss. The condition $\Gamma \subseteq \Lambda_N$ is satisfied from the definition of $\mathcal{Y}_t$ and Ψ_j, and the invariant condition is a consequence of the each Γ_j being a TRPI set.

Remark 5.7. *Considering compact polyhedral state and input constraints ($\mathcal{A}_N$ a polyhedral set), a natural choice of Ψ_j is given by $\Psi_j = \{(x, \theta) \in \mathbb{R}^{n+m} \colon \theta \in \Theta_j,\ (x, K_t x + \theta) \in \mathcal{A}_N\}$, where $\Theta_j \supseteq \theta(Y_j)$ is a polyhedral set. It is advantageous to consider relaxed constraints on θ, in an attempt to have $(\varrho_x(y_s), \theta(y_s)) \in \Phi_j$, and thus $(\varrho_x(y_s), y_s) \in \Gamma_j$, for all $y_s \in \mathcal{Y}_j$ [4]. However, the set Θ_j cannot be arbitrarily large, as discussed in [21, 30], otherwise the set Φ_j may not be finitely determined (Remark 2.3). Furthermore, as discussed in Section 2.6, additional constraints on state or input can be included on Ψ_j, in order to reduce the deviation between nonlinear and linearized models.*

Remark 5.8. *For simplicity, the same terminal control and cost matrices were considered for every partition $\mathcal{Y}_j$. Nonetheless, multiple pairs $({}^jK_t, {}^jP)$ can be defined, one for each*

[3]Notice that $(\varrho_x(y_s), \theta(y_s)) \in \hat{\Psi}$ for all $y_s \in \mathcal{Y}_j$, since $(\varrho_x(y_s), K_t\varrho_x(y_s) + \theta(y_s)) = (\varrho_x(y_s), \varrho_u(y_s))$ and $\mathcal{Y}_j \subseteq \mathcal{Y}_s(\mathcal{A}_N)$.

[4]Notice that if this condition is satisfied for all $j = 1 \dots N_p$, it is possible to directly choose $\mathcal{Y}_t = \hat{\mathcal{Y}}_t$.

$j = 1 \dots N_p$, with the TRPI sets being computed analogously. The terminal control law and cost would then be given respectively by $u_t(x, y_s) = K_t(y_s)(x - x_s) + u_s$ and $V_t(x, y_s) = (x - x_s)^\intercal P(y_s)(x - x_s)$, where $(K_t(y_s), P(y_s)) = ({}^jK_t, {}^jP)$ for $y_s \in \mathcal{Y}_j$.

These solutions were proposed in order to simplify the definition of the terminal cost and set. Nonetheless, the recursive feasibility and ISS guarantees presented in Section 5.2 are general and other stabilizing laws, terminal costs and TRPI sets can be considered in the proposed tracking NMPC algorithm.

Recapitulation

In this chapter, the robust NMPC presented in Chapter 2 was extended to follow piece-wise constant references, maintaining robust constraint satisfaction, recursive feasibility and input-to-state stability. In particular, the following topics were discussed:

- Equilibrium condition: The sets of admissible equilibrium points and reachable references were defined and the conditions under which each desired output is associated to a single steady-state were presented.

- Robust NMPC for tracking: The model predictive controller for tracking piece-wise constant references was presented. An artificial reference was inserted in the optimization problem to avoid feasibility loss during reference changes and the assumptions presented in Chapter 2 were extended to the tracking case. Under these modified assumptions, recursive feasibility and input-to-state stability of the closed-loop system were proven.

- Simplified terminal ingredients: Practical methods for choosing a quadratic terminal control law and polyhedral terminal robust positively invariant set, which satisfy the necessary assumptions, were provided.

Chapter 6

Stochastic Disturbances and Chance Constraints

In the previous chapters, NMPC control laws with constraints on state and input were presented and robust constraint satisfaction was ensured. This means that the system trajectory will satisfy the constraints for any disturbance realization. However, in order to guarantee robust constraint satisfaction, the worst-case disturbance realizations need to be considered, even if the probability of them actually occurring is remote, which may be rather conservative.

In this section, the additive disturbances are seen as stochastic variables and chance constraints, which allow for a predetermined level of admissible constraint violation, are considered. This is referred in the literature as Stochastic Model Predictive Control (SMPC) [34, 30] and avoids the conservativeness of always considering the worst-case disturbance scenario.

This approach reveals a *trade-off* between domain of attraction and performance, and admissible probability of constraint violation, where a larger domain of attraction and lower optimal costs can be achieved as long as a higher chance of constraint violation is allowed. In Section 6.1, chance constraints and their reformulation into *one-step-ahead* deterministic constraints are presented, while Section 6.2 shows how they can be incorporated into the NMPC strategies presented in the previous chapters, maintaining recursive feasibility and stability properties.

6.1 Chance Constraints

Consider system (2.1), where $w_k \in \mathcal{W}$ is a random variable with a given probability distribution with finite support. Individual chance constraints can then be imposed on

x_{k+1}, given the information available at k, as defined by

$$\mathbb{P}[h_j^{cc} x_{k+1} \leq g_j^{cc}] \geq 1 - \varepsilon_j, \quad j = 1 \dots n_c,\ \forall k \in \mathbb{N}, \tag{6.1}$$

where $h_j^{cc} \in \mathbb{R}^{1\times n}$, $g_j^{cc} \in \mathbb{R}$ and $\varepsilon_j > 0 \in \mathbb{R}$ define the linear inequalities and allowable probability of constraint violation for each chance constraint. Eq. (6.1) states that, given the information available at k, i.e. x_k and v_k, the probability that x_{k+1} satisfies the linear constraint $h_j^{cc} x_{k+1} \leq g_j^{cc}$ is at least $1 - \varepsilon_j$. Alternatively, ε_j is the maximum allowable probability of constraint violation.

The individual chance constraints can be converted into a deterministic constraint on the *one-step-ahead* prediction $x_{k+1|k} = f_\pi(x_k, v_k)$ as shown by Lemma 6.1, derived from [34].

Lemma 6.1 ([34]). *Consider system* (2.1)*, the line vector* $h \in \mathbb{R}^{1\times n}$*, the scalars* $g \in \mathbb{R}$*, and* $\varepsilon > 0 \in \mathbb{R}$*. Let* $\gamma_0 \in \mathbb{R}$ *be such that* $\mathbb{P}[hw_k \leq \gamma_0] \geq 1 - \varepsilon$*. Then, we have*

$$hx_{k+1|k} \leq g - \gamma_0 \ \Rightarrow\ \mathbb{P}[hx_{k+1} \leq g] \geq 1 - \varepsilon, \tag{6.2}$$

where $x_{k+1|k} = f_\pi(x_k, v_k)$ *is known at the time-instant* k*.*

Proof. We have $x_{k+1} = x_{k+1|k} + w_k$ and thus

$$hw_k \leq \gamma_0 \ \Rightarrow\ hx_{k+1} \leq hx_{k+1|k} + \gamma_0.$$

Therefore, if $hx_{k+1|k} \leq g - \gamma_0$, then $hw_k \leq \gamma_0$ implies $hx_{k+1} \leq g$ and $\mathbb{P}[hx_{k+1} \leq g] = \mathbb{P}[hw_k \leq \gamma_0] \geq 1 - \varepsilon$. □

Therefore, considering the chance constraint set given by

$$\mathcal{X}^{cc} = \{x \in \mathbb{R}^n \colon h_j^{cc} x \leq g_j^{cc} - \gamma_j,\ j = 1 \dots n_c\}, \tag{6.3}$$

where $\gamma \in \mathbb{R}^{n_c}$ satisfies $\mathbb{P}[h_j^{cc} w_k \leq \gamma_j] \geq 1 - \varepsilon_j,\ j = 1 \dots n_c$, then $x_{k+1|k} \in \mathcal{X}^{cc}$ implies $\mathbb{P}[h_j^{cc} x_{k+1} \leq g_j^{cc}] \geq 1 - \varepsilon_j,\ j = 1 \dots n_c$ and the set $\mathcal{X}^{cc}$ can be used to incorporate the chance constraints (6.1) into the NMPC optimization problem.

6.2 NMPC Algorithms with Chance Constraints

Similarly to the case with deterministic constraints, tightened constraints are considered in order to maintain recursive feasibility. Given the initial constraint $\mathcal{X}^{cc}(1) = \mathcal{X}^{cc}$ and disturbance propagation sets $\mathcal{S}(j)$ satisfying $x_{k+j+1|k+1} - x_{k+j+1|k} \in \mathcal{S}(j),\ j = 1 \dots N$, tightened chance constraint sets $\mathcal{X}^{cc}$ are iteratively given by:

$$\mathcal{X}^{cc}(j+1) = \mathcal{X}^{cc}(j) \ominus \mathcal{S}(j), \quad j = 1 \dots N-1. \tag{6.4}$$

Based on these sets, the following additional constraints, which implicitly ensure the chance constraints (6.1), can be included in the NMPC optimization problems:

$$x_{k+j|k} \in \mathcal{X}^{cc}(j), \quad j = 1 \dots N-1. \tag{6.5}$$

Remark 6.1. *In order to calculate the disturbance propagation sets $\mathcal{S}(j)$, $j = 1, ..., N$, the vector (x_k, v_k) should be bounded by a compact set. The constraint $(x_k, v_k) \in \mathcal{Z}_\pi$, $\forall k \in \mathbb{N}$ is a natural choice, but the interval matrix $\mathbf{J}_\pi$, and as a consequence the conservativeness of $\mathcal{S}(j)$, are potentially reduced from tighter constraints on (x_k, v_k).*

If chance constraints are used, notice that $x_k \in \mathcal{X}^{cc} \oplus \mathcal{W}$ for any $k \geq 1$, from the restriction $x_{k|k-1} \in \mathcal{X}^{cc}$ applied in the previous sampling instant. Thus, given a set $\mathcal{X}^o \supseteq \mathcal{X}^{cc} \oplus \mathcal{W}$ that satisfies $x_0 \in \mathcal{X}^o$ (the initial state is bounded by $\mathcal{X}^o$), $\mathbf{J}_\pi$ can be computed from the intersection $\mathcal{A}_0 = \mathcal{Z}_\pi \cap (\mathcal{X}^o \times \mathbb{R}^m)$. Notice that, with this remark, the condition of compactness of $\mathcal{Z}$ can be softened, since only compactness of $\mathcal{A}_0$ is needed (this allows, for example, cases where all state constraints are probabilistic).

In the following, for notation simplicity, the NMPC optimization problems $P_N(x_k)$, $P_N^\mu(x_k, \hat{\mu}_k)$ and $P_N^t(x_k, y_t)$ presented in Chapters 2, 3 and 5 with the addition of the constraints (6.5) will be represented by $\tilde{P}_N(x_k)$, $\tilde{P}_N^\mu(x_k, \hat{\mu}_k)$ and $\tilde{P}_N^t(x_k, y_t)$, respectively, and the ensuing NMPC control laws by $u_k = \tilde{\kappa}(x_k)$, $u_k = \tilde{\kappa}_\mu(x_k, \hat{\mu}_k)$ and $u_k = \tilde{\kappa}_r(x_k, y_t)$.

Finally, the concept of admissibility of the terminal set must be adapted in order to consider the presence of chance constraints, as shown by the following assumption:

Assumption 6.1. *The admissible N-step-ahead set $\mathcal{A}_N \subset \mathbb{R}^{n \times m}$ must satisfy:*

$$\mathcal{A}_N \subseteq \{(x, \pi(x, v)) \in \mathbb{R}^{n+m} \colon (x, v) \in \mathcal{Z}_\pi(N), x \in \mathcal{X}^{cc}(N)\}, \tag{6.6}$$

where in the regulation case $\mathcal{X}_f \subseteq \mathcal{V}_N = \{x \in \mathbb{R}^n \colon (x, u_t(x)) \in \mathcal{A}_N\}$ and in the tracking case $\Gamma \subseteq \Lambda_N = \{(x, y) \in \mathbb{R}^n \times \mathcal{Y}_t \colon (x, u_t(x, y)) \in \mathcal{A}_N\}$.

Now, Lemma 6.2 and Theorem 6.1 show that the recursive feasibility and stability guarantees remain in the new stochastic NMPC control laws, introducing the additional constraints (6.5) and Assumption 6.1.

Lemma 6.2 (Recursive Feasibility)**.** *The optimization problems $\tilde{P}_N(x_k)$, $\tilde{P}_N^\mu(x_k, \hat{\mu}_k)$ and $\tilde{P}_N^t(x_k, y_t)$, under the additional Assumption 6.1, are recursively feasible. In particular:*

(i) If $\hat{\mathbf{v}}^*(x_k) = (v_0^*, \dots, v_{N-1}^*)$ is a solution of $\tilde{P}_N(x_k)$, then $\hat{\mathbf{v}}^c = (v_1^*, \dots, v_{N-1}^*, v_t(x_{k+N|k}^*))$ defines a feasible solution of $\tilde{P}_N(x_{k+1})$.

(ii) If $\hat{\mathbf{v}}^*(x_k) = (v_0^*, \dots, v_{N-1}^*)$ is a solution of $\tilde{P}_N^\mu(x_k, \hat{\mu}_k)$ then $\hat{\mathbf{v}}^c = (v_1^*, \dots, v_{N-1}^*, v_t(x_{k+N|k}^*))$ defines a feasible solution of $\tilde{P}_N^\mu(x_{k+1}, \hat{\mu}_{k+1})$.

(iii) If $\hat{\mathbf{v}}^*(x_k, y_t) = (v_0^*, v_1^*, \ldots, v_{N-1}^*)$ *and* $y_s^*(x_k, y_t) = y_s^*$ *are a solution of* $\tilde{P}_N^t(x_k, y_t)$*, then* $\hat{\mathbf{v}}^c = (v_1^*, \ldots, v_{N-1}^*, v_t(x_{k+N|k}, y_s^*))$ *and* $y_s^c = y_s^*$ *define a feasible solution of* $P_N^t(x_{k+1}, \overline{y}_t)$*, for any* $\overline{y}_t \in \mathbb{R}^p$*.*

Proof. The proof is divided into the three cases, and the same notations applied in the proofs of Lemmas 2.1, 3.1 and 5.1 are used. Notice that, from the definition of the disturbance propagation sets, we have:

$$x^c_{k+j|k+1} \in x^*_{k+j|k} \oplus \mathcal{S}(j-1), \quad j = 1 \ldots N \tag{6.7}$$

(i) The constraints $(x^*_{k+j|k}, v^*_{k+j|k}) \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1,\ x^*_{k+j|k} \in \mathcal{X}^{cc}(j),\ j = 1 \ldots N-1$ and $x^*_{k+N|k} \in \mathcal{X}_f$ imply that: $(x^c_{k+1+j|k+1}, v^c_{k+1+j|k+1}) \in (x^*_{k+1+j|k}, v^*_{k+1+j|k}) \oplus \{\mathcal{S}(j) \times 0\} \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1$ and $x^c_{k+1+j|k+1} \in x^*_{k+1+j|k} \oplus \mathcal{S}(j) \in \mathcal{X}^{cc}(j),\ j = 1 \ldots N-1$, from (6.7) and the admissibility of $\mathcal{X}_f$ ($\mathcal{X}_f \subseteq \mathcal{V}_N$).

For the terminal constraint, since $x^c_{k+1+N|k+1} = f_\pi(x^c_{k+N|k+1}, v_t(x^*_{k+N|k}))$ and $x^c_{k+N|k+1} - x^*_{k+N|k} \in \mathcal{S}(N-1)$, $x^c_{k+1+N|k+1} \in f_\pi(x^*_{k+N|k}, v_t(x^*_{k+N|k})) \oplus \mathcal{S}(N)$ and, through the robust invariance of $\mathcal{X}_f$, $x^c_{k+1+N|k+1} \in \mathcal{X}_f$. Therefore, $\hat{\mathbf{v}}^c$ is a feasible candidate solution of $\tilde{P}_N(x_{k+1})$.

(ii) Once again, from (6.7) and $\mathcal{X}_f \subseteq \mathcal{V}_N$, $(x^*_{k+j|k}, v^*_{k+j|k}) \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1$, $x^*_{k+j|k} \in \mathcal{X}^{cc}(j),\ j = 1 \ldots N-1$ and $x^*_{k+N|k} \in \mathcal{X}_f$ imply $(x^c_{k+1+j|k+1}, v^c_{k+1+j|k+1}) \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1$ and $x^c_{k+1+j|k+1} \in \mathcal{X}^{cc}(j),\ j = 1 \ldots N-1$.

For the terminal constraint, $x^c_{k+1+N|k+1} = f_\pi(x^c_{k+N|k+1}, v_t(x^*_{k+N|k})) + \hat{\mu}_{k+1} \in f_\pi(x^*_{k+N|k}, v_t(x^*_{k+N|k})) + \hat{\mu}_{k+1} \oplus \mathcal{S}(N)$. Hence, from the positive invariance of $\mathcal{X}_f$, $x^c_{k+1+N|k+1} \in \mathcal{X}_f$. Therefore, $\hat{\mathbf{v}}^c$ is a feasible candidate solution of $\tilde{P}_N^\mu(x_{k+1}, \hat{\mu}_{k+1})$.

(iii) The constraints $(x^*_{k+j|k}, v^*_{k+j|k}) \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1,\ x^*_{k+j|k} \in \mathcal{X}^{cc}(j),\ j = 1 \ldots N-1$ and $(x^*_{k+N|k}, y^*_{s,k}) \in \Gamma$ imply, $(x^c_{k+1+j|k+1}, v^c_{k+1+j|k+1}) \in \mathcal{Z}_\pi(j),\ j = 0 \ldots N-1$ and $x^c_{k+1+j|k+1} \in \mathcal{X}^{cc}(j),\ j = 1 \ldots N-1$, from (6.7) and $\Gamma \subseteq \Lambda_N$.

For the terminal constraint, $x^c_{k+1+N|k+1} = f_\pi(x^c_{k+N|k+1}, v_t(x^*_{k+N|k}, y_s)) \in f_\pi(x^*_{k+N|k}, v_t(x^*_{k+N|k}, y_s)) \oplus \mathcal{S}(N)$ implies $(x^c_{k+1+N|k+1}, y_s) \in \Gamma$, since Γ is a robust positive invariant set for tracking. Therefore, $\hat{\mathbf{v}}^c$, $y_s^c = y^*_{s,k}$ define a feasible candidate solution of $\tilde{P}_N^t(x_{k+1}, \overline{y}_t)$, for any $\overline{y}_t \in \mathbb{R}^p$.

□

Theorem 6.1 (Input-to-State Stability)**.** *The stochastic NMPC control laws* $u_k = \tilde{\kappa}(x_k)$*,* $u_k = \tilde{\kappa}_\mu(x_k, \hat{\mu}_k)$ *and* $u_k = \tilde{\kappa}_r(x_k, y_t)$ *are Input-to-State Stable. In particular:*

(i) System (2.1) *subject to the NMPC control law* $u_k = \tilde{\kappa}(x_k)$ *satisfies:*

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|\mathbf{w}_{[0,k]}\|). \tag{6.8}$$

(ii) System (2.1) *subject to the NMPC control law* $u_k = \tilde{\kappa}_\mu(x_k, \hat{\mu}_k)$ *satisfies:*

$$\left\|x_k - \hat{x}^{\mu}_{0,k}\right\| \leq \beta(\left\|x_0 - \hat{x}^{\mu}_{0,0}\right\|, k) + \gamma(\left\|\mathbf{w}_{[0,k]}\right\|). \tag{6.9}$$

(iii) System (2.1) *subject to the NMPC control law* $u_k = \tilde{\kappa}_r(x_k, y_t)$*, where* $y_t \in \mathbb{R}$ *is a constant reference, satisfies:*

$$\|x_k - x^o_s\| \leq \beta\left(\|x_0 - x^o_s\|, k\right) + \gamma(\left\|\mathbf{w}_{[0,k]}\right\|), \tag{6.10}$$

where β *and* γ *represent appropriate* $\mathcal{KL}$*- and* $\mathcal{K}$*-functions, respectively.*

Proof. Analogous to the proofs of Theorems 2.1, 3.1 and 5.1, using Lemma 6.2 to ensure feasibility of the *one-step-ahead* candidate solution and Lemma B.6 to guarantee admissibility of the terminal control law inside the terminal set. □

Recapitulation

In this chapter, it was shown how to incorporate chance constraints into the predictive controllers previously presented, maintaining the recursive feasibility and stability guarantees. In particular, the following topics were discussed:

- Chance constraints: Chance state constraints, in the form of a minimal probability of constraint satisfaction by the next state given current information, were stated. It was also shown how this chance constraint on x_{k+1} can be reformulated as a deterministic constraint on the *one-step-ahead* prediction $x_{k+1|k}$.

- NMPC algorithms with chance constraints: The chance state constraints were then tightened via the disturbance propagation sets and incorporated into the controller design, such as to maintain the recursive feasibility and input-to-state stability guarantees.

Chapter 7

Case Studies

In this chapter, simulations are presented in order to validate the performance of the proposed NMPC algorithms. The first case study applies the robust NMPC with disturbance propagation via zonotopes presented in Chapter 2 to the DC-DC Buck-Boost converter [17, 34], comparing the zonotopic method of disturbance propagation proposed to the one based on Lipschitz constants. The second case study considers the robust NMPC for tracking presented in Chapter 5, with chance state constraints (Chapter 6). Piece-wise constant reference tracking and probability of constraint violation under the specified maximum are verified.

The next case studies are based on the CSTR (Continually Stirred Tank Reactor) benchmark, which consists of a tank used to perform an exothermic irreversible reaction [23, 24]. First, reference correction and the constant disturbance model are incorporated in the controller project, as proposed in Chapter 3, demonstrating that regulation without offset can then be achieved in the presence of constant disturbances. Finally, the tracking NMPC is also implemented in this case study, showing the importance of the artificial reference in preventing feasibility loss during reference changes and the increased domain of attraction provided by this strategy.

All simulations were made in an i7, 2.4 GHz, 16 GB RAM, DELL computer.

7.1 Buck-Boost Converter

The discrete-time nonlinear model of the Buck-Boost converter, with the equilibrium translated to the origin, can be represented by (2.1), with

$$
\begin{aligned}
f(x,u) &= \begin{pmatrix} x_1 + \alpha_1 x_2 + (\beta_1 - \gamma_2 x_2)\, u \\ -\alpha_2 x_1 + \alpha_3 x_2 + (\beta_2 + \gamma_1 x_1)\, u \end{pmatrix}, \\
h(x,u) &= x_2,
\end{aligned} \tag{7.1}
$$

where x_1, x_2 and u represent, respectively, the inductor current, output tension and *duty-cycle* input of the converter, translated in relation to an equilibrium point. For a sampling period $T_s = 0.65ms$, circuit parameters $R = 85\Omega$, $C = 2.2mC$, $L = 4.2mH$, $V_{in} = 15V$, and the equilibrium point defined by $V_{out} = -16V$, the model parameters are: $\alpha_1 = 0.07488\Omega^{-1}$, $\alpha_2 = 0.1430\Omega$, $\alpha_3 = 0.9965$, $\beta_1 = 4.798A$, $\beta_2 = 0.1149V$, $\gamma_1 = 0.2955\Omega$ and $\gamma_2 = 0.1548\Omega^{-1}$. The following state and input constraints were considered:

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^2 \colon \|x\|_\infty \leq 3\}, \\ \mathcal{U} &= \{u \in \mathbb{R} \colon |u| \leq 0.3\}, \end{aligned} \tag{7.2}$$

and the system is subject to additive disturbances limited by the box $\mathcal{W} = \{w \in \mathbb{R}^2 \colon \|w\|_\infty \leq 0.04\}$.

7.1.1 Regulation

In this section, the Buck-Boost system is used in order to compare the disturbance propagation strategies based on zonotopes and lipschitz constants, and the corresponding NMPC strategies with constraint tightening.

Through the algorithm proposed in Appendix A, for this particular case we have $K_v = \begin{pmatrix} 0 & 0 \end{pmatrix}$, with associated Lipschitz constant $L_x = 1.228$, and thus $u_k = v_k$ was directly applied.[1] For the NMPC design parameters, a prediction horizon of $N = 4$ and, following the simplifying assumptions of Section 2.5, a cost function $L_\pi(x, v) = x^\intercal Q x + u^\intercal R u$, with $Q = I$ and $R = 1$, were chosen. The stabilizing terminal control law and associated terminal cost can then be obtained from Theorem 2.2 via the method proposed in [15], resulting in $K_t = \begin{pmatrix} -0.2534 & 0.3150 \end{pmatrix}$ and $P = \begin{pmatrix} 3.398 & -5.079 \\ -5.079 & 27.67 \end{pmatrix}$.

Initially, in order to evaluate the effect of the zonotopic approach in conservatism reduction of the disturbance propagation sets, the sets $\mathcal{S}(j)$ calculated via the zonotopic method (Property 4.1) and via Lipschitz constants (4.1) are compared.

For comparison purposes and in order to verify the Condition 2.1 satisfaction, nominal trajectories of system (7.1), with $u = K_v x = 0$, were simulated for a grid of points in the set $x_0 \oplus \mathcal{W}$, where x_0 is a point in $\mathcal{X}$ such that the ensuing trajectories satisfy state and input constraints. The result is shown in Figure 7.1.

As expected, both the zonotopes $x_{j|0} \oplus \mathcal{S}_z(j)$ and the boxes $x_{j|0} \oplus \mathcal{S}_l(j)$ contain the nominal trajectories of all points in $x_0 \oplus \mathcal{W}$ (Condition 2.1). The sets $\mathcal{S}_z(j)$, however, are contained (Corollary 4.1) and are considerably smaller than the $\mathcal{S}_l(j)$, specially for

[1] The prediction feedback matrix $K_v = 0$ because in this case the interval matrix $\mathbf{J}_u$, which depends on the states, has a considerably larger radius than $\mathbf{J}_x$, which depends on the input. Therefore, to minimize the Lipschitz constant, it is better for this particular system to directly make $\mathbf{J}_\pi = \mathbf{J}_x$.

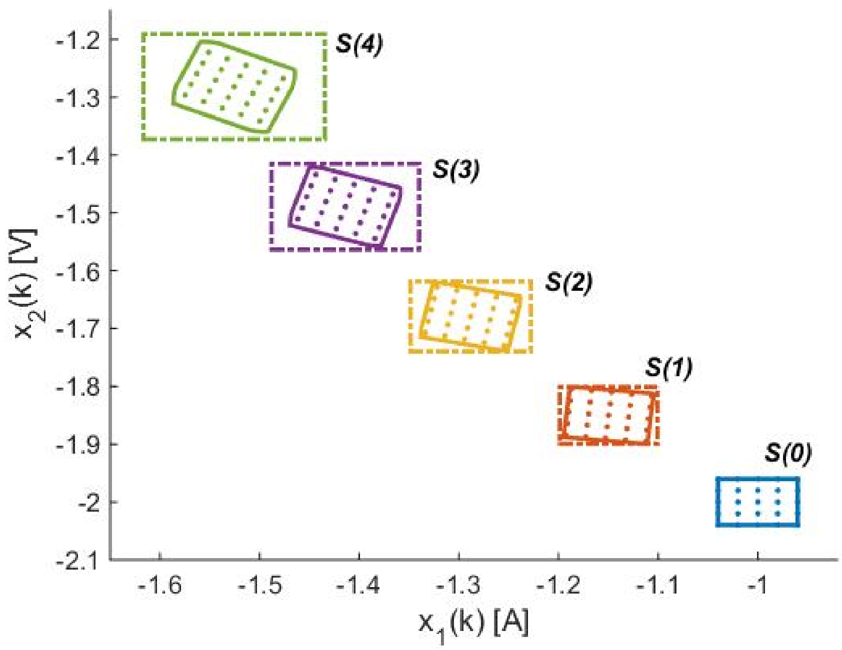


Figure 7.1: Comparison of the sets $\mathcal{S}(j)$ via zonotopic and Lipschitz methods. The sets $\mathcal{S}_z(j)$ are represented by solid lines, while the $\mathcal{S}_l(j)$ by dashed lines.

larger values of j (longer prediction horizons). Table 7.1 compares the sizes of the sets, represented by their areas, and the computation time necessary for their calculation (t_c).[2] Notice that the zonotopic approach offers less conservative limits for these trajectories, still with a low computational cost[3], and thus better estimates the disturbance propagation.

Table 7.1: Disturbance Propagation Sets Comparison (Buck-Boost).

Size($\times 10^{-3}$ VA)/Comp. Cost(μs)	$\mathcal{S}(0)$	$\mathcal{S}(1)$	$\mathcal{S}(2)$	$\mathcal{S}(3)$	$\mathcal{S}(4)$	t_c
Zonotopic Method	6.40	7.42	8.77	10.5	12.7	244
Lipschitz Method	6.40	9.65	14.6	22.0	33.1	55

Robust model predictive controllers were then implemented via the method described in Chapter 2, one applying the zonotopes $\mathcal{S}_z(j)$ and other the boxes $\mathcal{S}_l(j)$ for the constraint tightening. Figure 7.2 compares the terminal sets $\mathcal{X}_f$, obtained via the method proposed in Section 2.6, and the domains of attraction of each controller. The closed-loop trajectories of both controllers were also simulated from the initial state $x_0 = (-1.5, -3)$, with simulation time $N_{sim} = 40$ and the same sequence of aleatory

[2]For a better comparison of computational costs, the cost associated to the computation of the interval matrices $\mathbf{J}_x$ and $\mathbf{J}_u$ and the feedback matrix K_v, which is identical for both methods, is not considered.

[3]It is worth noting that, since the sets $\mathcal{S}(j)$ are computed offline, the online computational cost of solving the NMPC optimization problem is the same for both strategies.

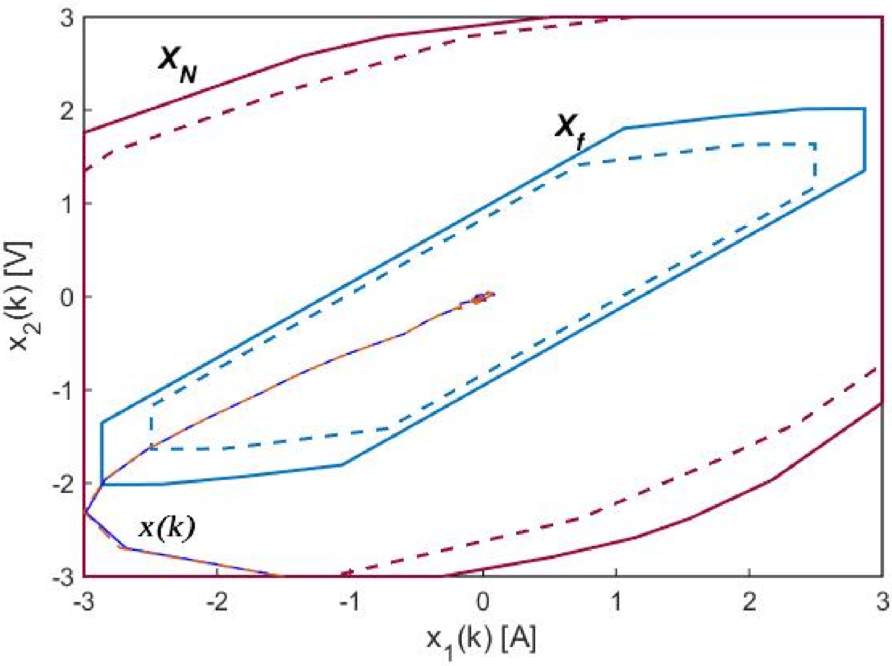


Figure 7.2: Comparison of terminal sets, domains of attraction and trajectories of the predictive controllers. Once again, results related to the zonotopic and Lipschitz methods are represented by solid and dashed lines, respectively.

disturbances $\mathbf{w}_{[0,N_{sim}-1]} \in \mathcal{W}^{N_{sim}}$.[4]

In summary, the proposed NMPC strategy is capable of reducing the conservatism in the computation of the tighter constraints through a zonotopic representation with low computational cost, such that the domain of attraction of the resulting controller is increased.

7.1.2 Chance Constraints

In this section, the Buck-Boost case study is used to illustrate the properties of the robust NMPC controller for tracking presented in Chapter 5 in the presence of chance constraints (Chapter 6).

The deterministic state constraints $x_k \in \mathcal{X} = \{x \in \mathbb{R}^2 \colon \|x\|_\infty \leq 3\}$ are thus replaced by individual chance constraints, defined by:

$$\begin{aligned} &\mathbb{P}[[x_{k+1}]_1 \leq 3] \geq 0.8, && \mathbb{P}[[x_{k+1}]_1 \geq -3] \geq 0.8, \\ &\mathbb{P}[[x_{k+1}]_2 \leq 3] \geq 0.8, && \mathbb{P}[[x_{k+1}]_2 \geq -3] \geq 0.8. \end{aligned} \tag{7.3}$$

The additive disturbances $w_k \in \mathcal{W} = \{w \in \mathbb{R}^2 \colon \|w\|_\infty \leq 0.04\}$ are derived from

[4]The sequence of disturbances has an uniform distribution on $\mathcal{W}$ and was generated by the *Mersenne Twister* with unitary seed.

a truncated normal distribution $\mathcal{N}(0, 0.02^2 I)$ and, as proposed in Remark 6.1, an outer bound on x_0, given by $\mathcal{X}^o = \{x \in \mathbb{R}^2\colon \ \|x\|_\infty \leq 3.2\}$, was considered for the computation of the disturbance propagation sets $\mathcal{S}(j)$.

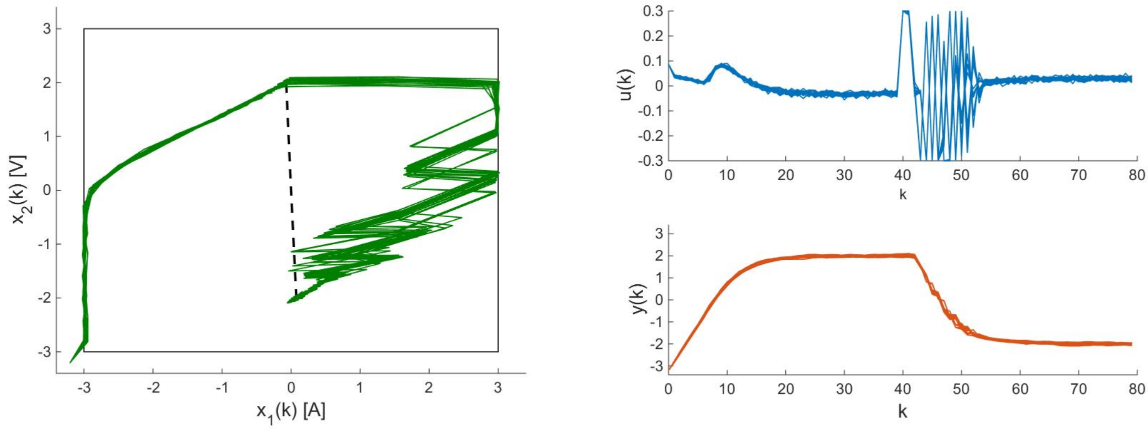
Again, the algorithm proposed in Appendix A results in $K_v = \begin{pmatrix} 0 & 0 \end{pmatrix}$. For the NMPC design parameters, a prediction horizon of $N = 4$ and quadratic stage and offset costs were chosen, with $L(x,u) = x^\intercal Q x + u^\intercal R u$, $Q = I$, $R = 1$, and $V_O(y) = y^\intercal T y$, $T = 1000$. The terminal control law $u = u_s + K_t(x - x_s)$ and terminal cost $V_f(x, y_s) = x^\intercal P x$, with $K_t = \begin{pmatrix} -0.2534 & 0.3150 \end{pmatrix}$ and $P = \begin{pmatrix} 3.398 & -5.079 \\ -5.079 & 27.67 \end{pmatrix}$, were obtained from Theorem 5.2 via the method proposed in [15].

For the computation of the TRPI set, the desired feasible equilibria set $\hat{\mathcal{Y}}_t = \{y \in \mathbb{R}\colon \ -2 \leq y \leq 2\}$ was partitioned into the disjoint sets: $\mathcal{Y}_1 = [-2, -1.5[$, $\mathcal{Y}_2 = [-1.5, 0[$, $\mathcal{Y}_3 = [0, 1.5[$ and $\mathcal{Y}_4 = [1.5, 2[$. From these partitions, four TRPI sets Γ_j were computed, as proposed in Section 5.3, satisfying $(\varrho_x(y_s), y_s) \in \Gamma_j, \ \forall y_s \in \mathcal{Y}_j, \ j = 1 \ldots 4$. Finally, $\mathcal{Y}_t = \hat{\mathcal{Y}}_t$ was chosen, with the terminal set given by $\Gamma = \bigcup_j \Gamma_j$.

The closed loop system was then simulated with piece-wise constant reference $y_{r,k} = 2, \forall k \leq 40$, $y_{r,k} = -2, \forall k > 40$ and initial state $x_0 = (-3.2, -3.2)$, for 300 disturbance realizations. The state-space trajectories, input and output responses of the first 20 simulations are shown in Figure 7.3. Notice that, as expected, the trajectories are steered to RPI sets around x_s^o. Furthermore, in order to evaluate the effect of the individual chance constraints and verify that the defined admissible constraint violation probabilities are indeed satisfied, the Empirical Cumulative Distribution Functions (ECDFs) of $[x_{2:5}]_1$ and $[x_{43:44}]_1$ [5] are shown in Figure 7.4. Notice that in both cases a probability of constraint violation under 20% was achieved (more than 80% of the values obtained satisfied the restrictions), and thus the individual chance constraints (7.3) are satisfied.

This result is expected, being similar to the ones obtained in the linear and regulation cases [34, 33]. However, to the best of the author's knowledge, there are no similar results which apply chance constraints to the tracking NMPC problem with an artificial reference.

[5] $[x_{a:b}]_1$ refers to the first coordinates of $x_a, \ldots, x_b$, for every realization



(a) (b)

Figure 7.3: (a) Phase-plane evolution of the closed-loop NMPC control system for 20 disturbance realizations (the dashed line represents the equilibrium states $\mathcal{X}_t = \varrho_x(\mathcal{Y}_t)$). (b) Input and Output time responses of the closed-loop NMPC control system for 20 disturbance realizations.

7.2 CSTR

The following nonlinear continuous-time model describes the CSTR (Continually Stirred Tank Reactor) system:

$$\begin{aligned}\dot{C}_A(t) &= \frac{\overline{q}}{V}(\overline{C}_{Af} - C_A(t)) - k_0 e^{-\frac{T_o}{T(t)}} C_A(t),\\ \dot{T}(t) &= \frac{\overline{q}}{V}(\overline{T}_f - T(t)) + \frac{-\Delta H_r}{\rho C_p} k_0 e^{-\frac{T_o}{T(t)}} C_A(t)\\ &\quad + \frac{UA}{V\rho C_p}(T_c(t) - T(t)),\end{aligned} \tag{7.4}$$

where the concentration of product A in the tank $C_A(t)$ and the reaction temperature $T(t)$ are the state variables, with the cooling temperature $T_c(t)$ as the control input. The reaction temperature is considered as the controlled output and the model parameters are: $\rho = 1000g/\ell$, $k_0 = 7.2 \times 10^{10} min^{-1}$, $UA = 5 \times 10^4 J/(minK)$, $T_o = 8750K$, $-\Delta H_r = 5 \times 10^4 J/mol$, $C_p = 0.239J/(gK)$, $V = 100\ell$, $\overline{q} = 100\ell/min$, $\overline{T}_f = 350K$ and $\overline{C}_{Af} = 1.0mol/\ell$.

The states and input are translated and scaled in order to simplify calculations and position the equilibrium point defined by $C_A^o = 0.5mol/\ell$, $T^o = 350K$ and $T_c^o = 300K$ in the origin as follows:

$$\begin{aligned} x_1(t) = \frac{C_A(t) - 0.5}{0.05}, \quad x_2(t) &= \frac{T(t) - 350}{2},\\ u(t) &= \frac{T_c(t) - 300}{20}.\end{aligned} \tag{7.5}$$

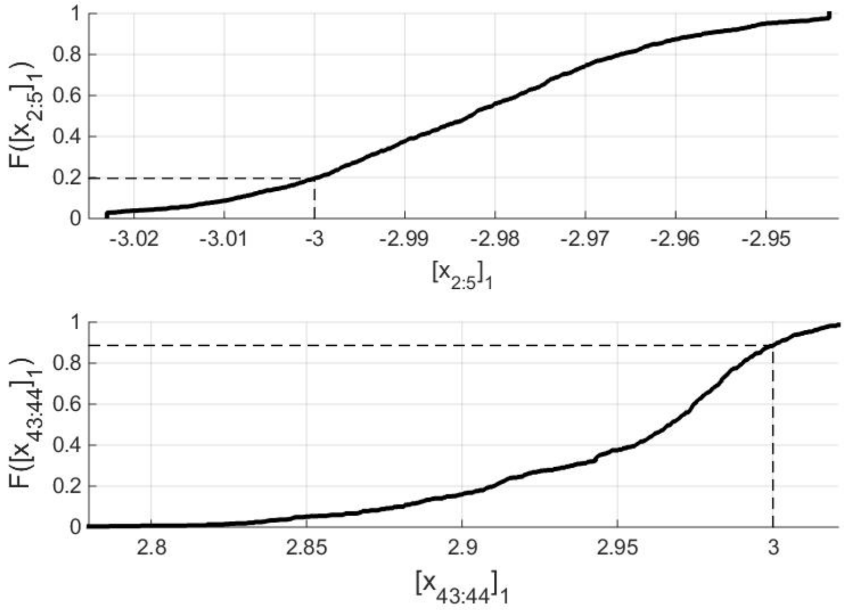


Figure 7.4: Empirical Cumulative Distribution Functions of $[x_{2:5}]_1$ and $[x_{43:44}]_1$ obtained from 300 simulations.

Therefore, the discrete-time nonlinear system given by (7.6) is considered.

$$
\begin{aligned}
x_{k+1} &= f_{CSTR}(x_k, u_k) + w_k, \\
y_k &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_k,
\end{aligned} \tag{7.6}
$$

where $f_{CSTR}\colon \mathbb{R}^3 \to \mathbb{R}^2$ is obtained via Euler discretization of system (7.4), with time interval $T_s = 0.03\ min$. The following constraints are considered: $0.1 mol/\ell \leq C_A(t) \leq 0.9 mol/\ell$, $340K \leq T(t) \leq 360K$ and $260K \leq T_c(t) \leq 340K$, which are converted by (7.5) to the following state and input constraint sets:

$$
\begin{aligned}
\mathcal{X} &= \{x \in \mathbb{R}^2\colon |x_1| \leq 8,\ |x_2| \leq 5\}, \\
\mathcal{U} &= \{u \in \mathbb{R}\colon |u| \leq 2\}.
\end{aligned} \tag{7.7}
$$

For the simulation of the tracking NMPC, additive disturbances limited by the box $\mathcal{W} = \{w \in \mathbb{R}^2\colon \|w\|_\infty \leq 0.1\}$ are considered, while for the presentation of the constant disturbance attenuation method a smaller disturbance set $\mathcal{W} = \{w \in \mathbb{R}^2\colon \|w\|_\infty \leq 0.05\}$ is assumed, due to the more conservative disturbance propagation associated with the incorporation of the mean-value disturbance estimates into the prediction.

7.2.1 Constant Disturbance Attenuation

In this Section, the CSTR system is used in order to illustrate the NMPC algorithm with constant disturbance attenuation presented in Chapter 3 and compare it to the original controller of Chapter 2, showing its benefits and drawbacks.

Regulation to the equilibrium defined by $C_A^o = 0.5mol/\ell$, $T^o = 350K$ and $T_c^o = 300K$, translated to the origin, is considered, and the mean-value estimates $\hat{\mu}_k$ are computed via a low-pass first order filter with pole $a = 0.9$ and unitary gain ($F(z) = \frac{(1-a)z}{z-a}$). The auxiliary sets $\overline{\mathcal{W}}$, $\mathcal{M}$ and $\mathcal{DM}$ are obtained from $\mathcal{W} = \{w \in \mathbb{R}^2\colon \|w\|_\infty \leq 0.05\}$ and the filter transfer function $F(z)$ as detailed in Section 3.2. Notice that an increase in the value of $a \in (0,1)$ results in a reduction of the bandwidth and the set $\mathcal{DM}$, however increases the settling time. Therefore, a trade-off between the convergence rate of the filtered estimation and smoothness of the target and prediction model correction is observed.

Through the algorithm proposed in Appendix A, the feedback prediction matrix K_v is given by $K_v = \begin{pmatrix} -0.3193 & -2.119 \end{pmatrix}$, with an associated infinity-norm Lipschitz constant $L_x = 1.102$. For the controller design, a prediction horizon of $N = 4$ and a quadratic stage-cost $L_\pi(x,v) = x^\intercal Q x + u^\intercal R u$, with $Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix}$ and $R = 5$, were chosen. The terminal control law and cost were obtained from Theorem 2.2 via the method proposed in [15], resulting in $K_t = \begin{pmatrix} -0.3590 & -2.010 \end{pmatrix}$ and $P = \begin{pmatrix} 28.30 & 1.730 \\ 1.730 & 43.75 \end{pmatrix}$.

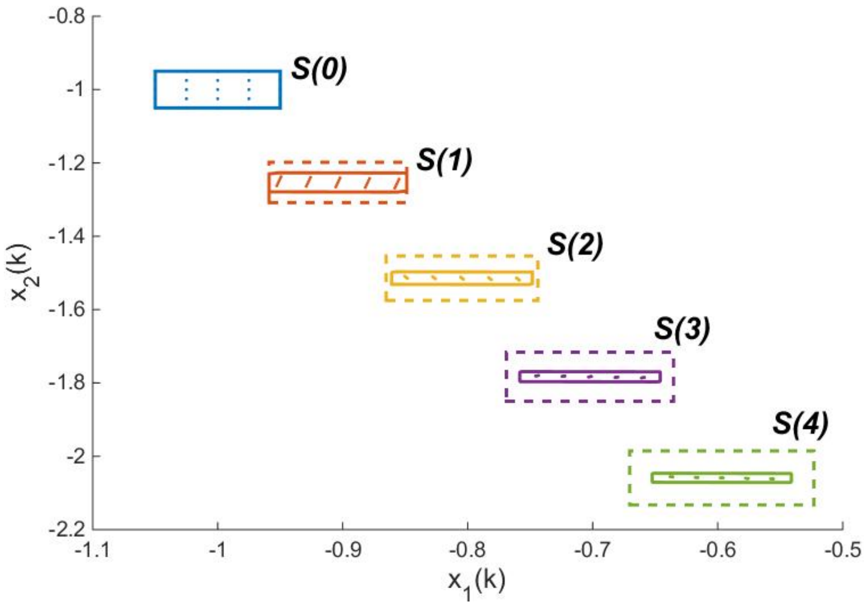
The disturbance propagation sets $\mathcal{S}(j)$ obtained via the zonotopic and Lipschitz methods, with and without the inclusion of the constant disturbance model in the prediction, were evaluated, with the results presented in Figure 7.5 and Table 7.2. Notice that the zonotopic methods are once again less conservative than the Lipschitz ones ($\mathcal{S}_z^0(j) \subseteq \mathcal{S}_l^0(j)$, $\mathcal{S}_z^\mu(j), \overline{\mathcal{S}_z^\mu}(j) \subseteq \mathcal{S}_l^\mu(j)$). In fact, in one direction the zonotopic methods are even able to reduce the size of the $\mathcal{S}(j)$, while the $\mathcal{S}_l(j)$ needs to increase by L_x in all directions.

Table 7.2: Disturbance Propagation Sets Comparison (CSTR).

Size($\times 10^{-3}\ Kmol/\ell$)/Comp. Cost($\mu s$)	$\mathcal{S}(0)$	$\mathcal{S}(1)$	$\mathcal{S}(2)$	$\mathcal{S}(3)$	$\mathcal{S}(4)$	t_c
$\mathcal{S}_z^0(j)$	10.0	5.71	3.84	3.07	2.74	176
$\mathcal{S}_l^0(j)$	10.0	12.1	14.7	17.9	21.7	50
$\mathcal{S}_z^\mu(j)$	40.0	29.8	26.4	26.6	28.5	450
$\overline{\mathcal{S}_z^\mu}(j)$	40.0	29.8	26.0	26.0	27.6	605
$\mathcal{S}_l^\mu(j)$	40.0	57.8	81.1	111.4	150.2	71

The sets $\mathcal{S}_z^\mu(j)$, $\overline{\mathcal{S}_z^\mu}(j)$ and $\mathcal{S}_l^\mu(j)$ are in general larger than the $\mathcal{S}_z^0(j)$ and $\mathcal{S}_l^0(j)$. This, however, is due to fact that the inclusion of the constant disturbance model requires the consideration of any $\hat{\mu}_k, \hat{\mu}_{k+1} \in \mathcal{M}$, $\hat{\mu}_{k+1} - \hat{\mu}_k = \mathcal{DM}$[6]. Therefore, the NMPC with

[6]Notice that the sets $\mathcal{S}_z^\mu(j)$, $\overline{\mathcal{S}_z^\mu}(j)$ and $\mathcal{S}_l^\mu(j)$ could be reduced via artificially limiting $\mathcal{M}$ and $\mathcal{DM}$.



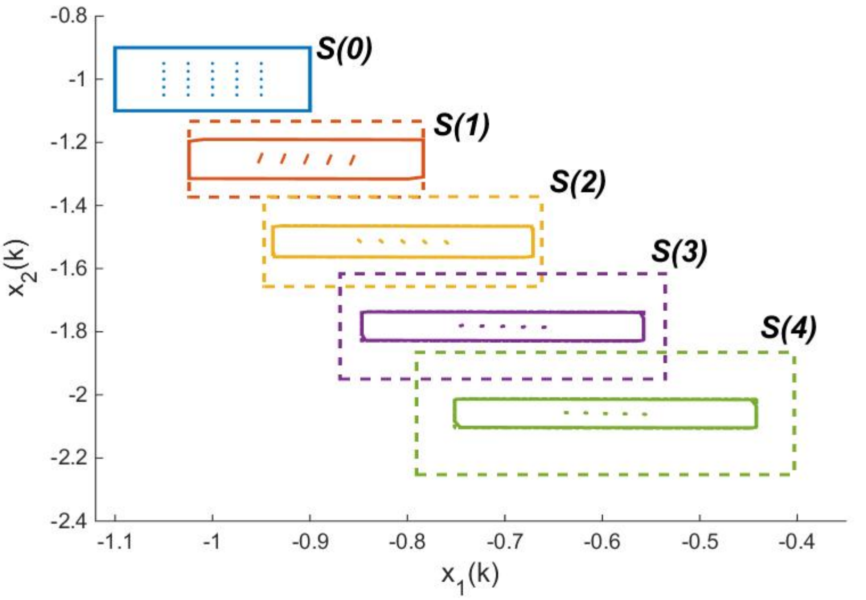


Figure 7.5: Comparison of disturbance propagation sets. The sets $\mathcal{S}_z^0(j)$, $\overline{\mathcal{S}_z^\mu}(j)$ are represented by solid lines, $\mathcal{S}_z^\mu(j)$ by dot-dashed lines and $\mathcal{S}_l^0(j)$, $\mathcal{S}_l^\mu(j)$ by dashed lines. Top: Nominal predictions. Bottom: Corrected predictions.

constant disturbance attenuation presents larger disturbance propagation sets and, as a consequence, a reduced domain of attraction. This is nonetheless necessary in order to guarantee robust constraint satisfaction and recursive feasibility despite the online actualization of the prediction model.

Furthermore, as shown by Table 7.2, the sets $\overline{\mathcal{S}_z^{\mu}}(j)$ are slightly less conservative than the $\mathcal{S}_z^{\mu}(j)$. This is in line with the discussion of Section 4.2 which led to the proposal of the alternative sets $\overline{\mathcal{S}_z^{\mu}}(j)$, i.e. the conservatism present in considering potentially different $\Delta\hat{\mu}(j) \in \mathcal{DM}$ for each disturbance propagation step. However, since the sets $\mathcal{S}_z^{\mu}(j)$ and $\overline{\mathcal{S}_z^{\mu}}(j)$ only differ in respect to the propagation of $\mathcal{DM}$, which tends to be considerably smaller than $\overline{\mathcal{W}}$, the difference is quite small.[7]

Finally, based on the zonotopic sets $\mathcal{S}_z^{0}(j)$ and $\mathcal{S}_z^{\mu}(j)$, NMPC controllers with and without constant disturbance attenuation were projected as described in Chapters 2 and 3. The controller responses were then simulated, with initial state $x_0 = (3, -3)$, simulation time $N_{sim} = 60$ and an additive disturbance consisting of a mean-value of $\mu_0 = (0.03, -0.02)$ added to a random zero-mean exponentially decreasing $\overline{w}_k$ (Figure 7.6). The closed loop responses are compared in Figure 7.7, while Table 7.3 presents the steady-state offset, Integral Absolute Error (IAE) and mean online computation time (t_c) of each controller.

As previously discussed, the BIBO low-pass filter is able to attenuate high-frequency variation such that the filtered disturbance converges to the constant steady-state value almost monotonically. This effect can be verified from the concentration target correction depicted in Figure 7.7. It should be remarked that the target correction is derived directly from the estimated disturbance. In this simulation, despite the noise effect, a smooth target correction is observed with respect to the concentration, which is a direct consequence of the high-frequency attenuation verified from the filtered estimation.

Notice that without constant disturbance attenuation the states tend to a disturbed equilibrium and the output is not regulated to the desired set-point. The algorithm proposed in Chapter 3, however, steers the states to the modified steady-state target $g_x(\mu_0)$, in such a way that $\lim_{k\to\infty} T(k) = 350$ as desired, despite the constant disturbance.

Table 7.3: Performance Comparison - Constant Disturbance Attenuation.

Measurement	Offset (K)	$IAE(K)$	$t_c(ms)$
Without CDA	0.14	33.0	121
With CDA	0	29.19	172

[7]This small difference can be seen from the similar areas shown in Table 7.2. Furthermore, the two sets are barely distinguishable in Figure 7.5.

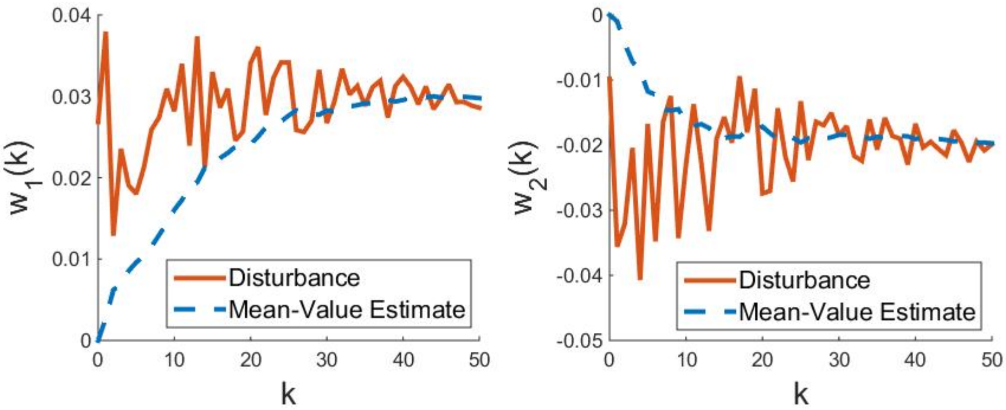


Figure 7.6: Sequence of disturbances $w(k)$, with associated mean-value estimates $\hat{\mu}(k)$.

Notice that the IAE is higher on the case without constant disturbance attenuation, and would in fact be unbounded for longer simulation times. There was a moderate increase on the computation time, although it is worth noting that this increase is due to the disturbance filtration and corrected steady-state calculation. The optimization problem, main source of computational stress (specially in more complex systems), has the same number of variables and constraints in both cases.

7.2.2 Tracking

In this Section, the robust NMPC algorithm for tracking presented in Chapter 5 is applied to the CSTR system, in order to validate the proposed controller and verify the influence of the artificial reference.

The matrix $K_v = \begin{pmatrix} -0.3193 & -2.119 \end{pmatrix}$ is obtained once again by the algorithm proposed in Appendix A, in order to mitigate the disturbance propagation. For the controller parameters, a prediction horizon of $N = 4$, with quadratic stage and offset costs, as proposed in Remark 5.2, were chosen, with $Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix}$, $R = 5$ and $T = 1000$. A linear terminal control law and quadratic terminal cost were then obtained by the method proposed in Section 5.3, resulting in $K_t = \begin{pmatrix} -0.3590 & -2.010 \end{pmatrix}$ and $P = \begin{pmatrix} 28.30 & 1.730 \\ 1.730 & 43.75 \end{pmatrix}$.

A terminal TRPI set was then computed as proposed in Section 5.3 from the desired set of feasible equilibria $\hat{\mathcal{Y}}_t = \{y \in \mathbb{R} \colon -4 \leq y \leq 4\}$, which was partitioned into the following disjoint sets: $\mathcal{Y}_1 = [-4, -3[$, $\mathcal{Y}_2 = [-3, -2[$, $\mathcal{Y}_3 = [-2, 0[$, $\mathcal{Y}_4 = [0, 2[$, $\mathcal{Y}_5 = [2, 3[$ and $\mathcal{Y}_6 = [3, 4]$. For illustration purposes, the projections on the phase plane of the six TRPI sets, $\Gamma_j,\ j = 1 \ldots 6$, obtained, as well as their corresponding feasible equilibria $\varrho_x(\mathcal{Y}_j)$, are presented in Figure 7.8. Since in this case $(\varrho_x(y_s), y_s) \in \Gamma_j,\ \forall y_s \in \mathcal{Y}_j$ was

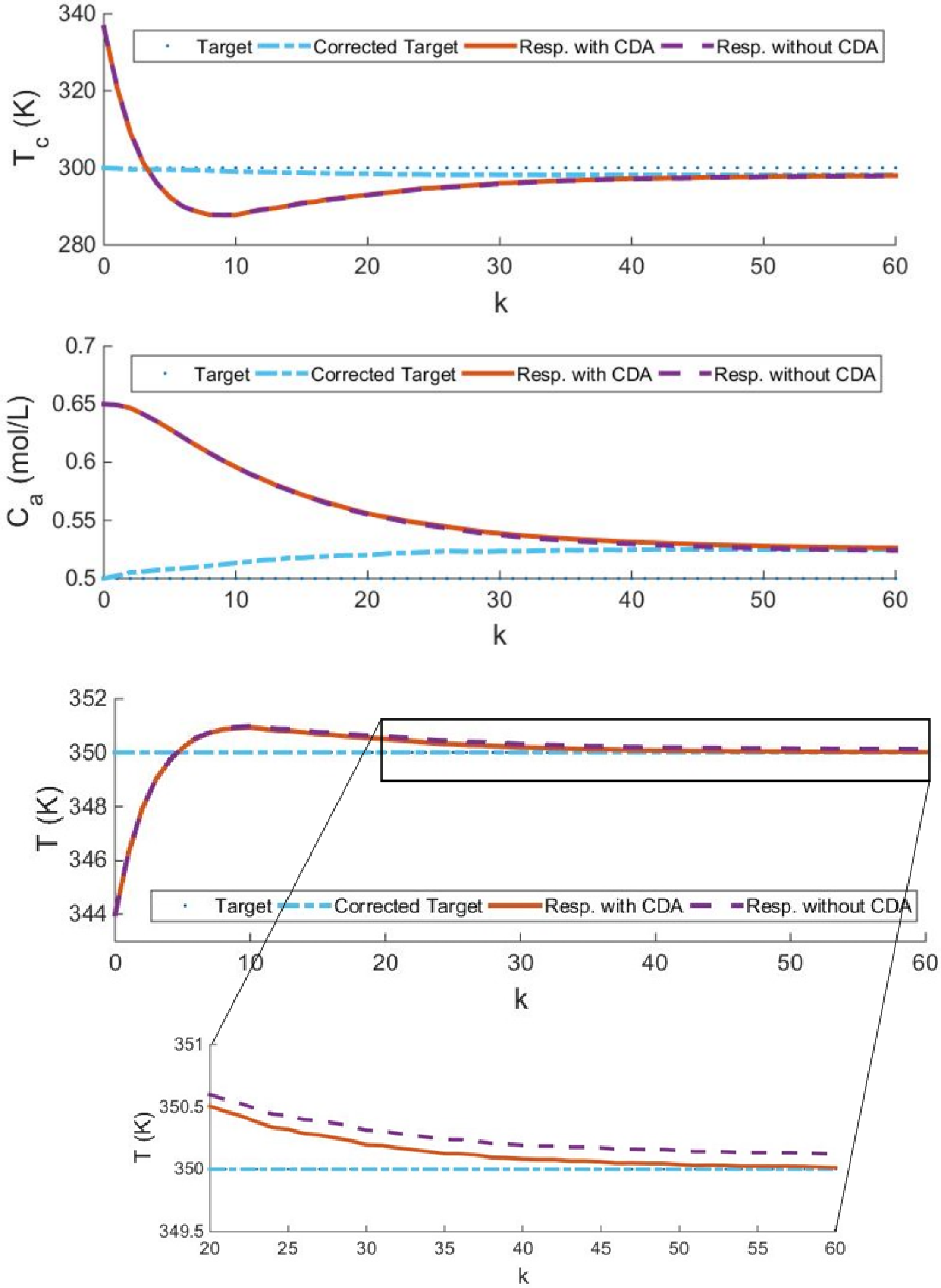


Figure 7.7: Comparison of closed-loop responses of NMPC controllers with and without Constant Disturbance Attenuation (CDA).

achieved through relaxed constraints on θ (Remark 5.7), $\mathcal{Y}_t = \hat{\mathcal{Y}}_t$ was chosen, with the terminal set given directly by $\Gamma = \bigcup_j \Gamma_j$.

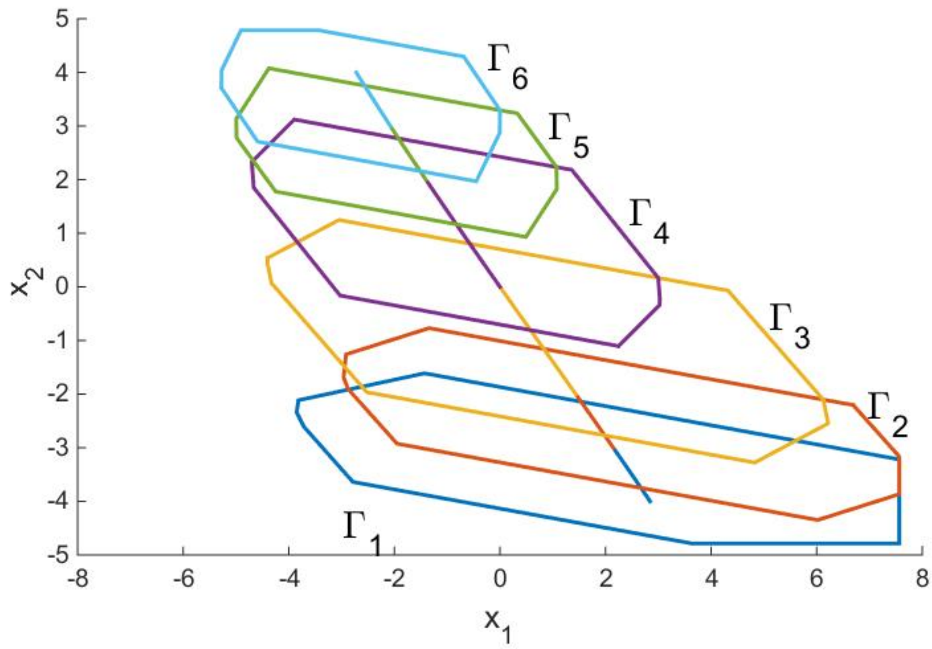


Figure 7.8: Projections on the phase plane of the TRPI sets Γ_j. The terminal set is given by the union $\Gamma = \bigcup_j \Gamma_j$.

The closed loop system with the NMPC control law (5.6) was then simulated with a piecewise constant reference for the nominal case and with a random sequence of disturbances $w_k \in \mathcal{W}$ [8], with the results given in Figures 7.9 and 7.10. Notice that in the nominal case the output converges asymptotically to the optimal admissible target y_s^o ($y_s^o = y_t$ in case of $y_t \in [-4, 4] = \mathcal{Y}_t$, $y_s^o = 4$ if $y_t > 4$ and $y_s^o = -4$ if $y_t < -4$), while in the presence of disturbances the state is steered to a RPI region around x_s^o.

The system was also simulated for other values of offset cost ($T = 20$ and $T = 10^4$), in order to evaluate its influence on performance. The resulting output and artificial reference responses are shown in Figure 7.11. Notice that the transient response of the artificial reference with respect to the optimal target $y_{s,k}^* \to y_s^o$ becomes slower with a smaller offset cost weighting, as $T = 20$, so that a longer overall output settling time is observed. If a sufficiently large T is used, as $T = 10^4$, the difference between the artificial and the the optimal target becomes negligible as soon as the optimal target provides a feasible solution for the optimization problem. This fast artificial target response comes from the relative impact of the offset cost with respect to the overall cost function. Indeed,

[8]The sequence of disturbances has an uniform distribution on $\mathcal{W}$ and was generated by the *Mersenne Twister* with unitary seed.

for an already sufficiently large offset cost, further increases on T have little influence on the controller response since, if $y_{s,k} = y_s^o$ is feasible, $y_{s,k}^*$ can be considered as virtually equal to y_s^o from a numerical approximation perspective. In this case, for example, $T = 1000$ already provides approximately the same response as $T = 10^4$ or any other $T > 10^4$ (or $T \to \infty$).

Finally, in order to illustrate the usefulness of the artificial reference, the NMPC for tracking was compared with a similar controller without the artificial reference (making $y_s = y_t$), which is equivalent to considering a NMPC for regulation translated to the reference y_t. Notice that in this case the feasibility of the optimization problem, and thus the domain of attraction, depends on the reference y_t. Figure 7.12 compares the domain of attraction of the NMPC for tracking with the domains of attraction for the regulation case, for $y_t = -3$ and $y_t = 3$.[9] As expected, due to the freedom provided by the artificial reference, the domain of attraction of the tracking NMPC contains the others. Furthermore, while in the tracking case any feasible equilibria is inside the domain of attraction, this is not the case without the artificial reference, where feasibility can be lost due to reference change (for example from $y_t = -3$ to $y_t = 3$).

Recapitulation

In this chapter, the robust NMPC algorithms were applied in simulation to the DC-DC Buck-Boost converter and CSTR benchmarks in order to validate and compare the proposed approaches. In particular, the following simulations were made:

- NMPC for regulation of the Buck-Boost converter: The NMPC for regulation presented in Chapter 2, with zonotopic disturbance propagation sets (Chapter 4), was applied in simulation to the Buck-Boost case-study. The zonotopic method was shown to be less conservative than the one based on Lipschitz constants, providing smaller disturbance propagation sets, and thus a greater domain of attraction.

- Stochastic NMPC of the Buck-Boost converter: The NMPC for tracking proposed in Chapter 5, with the incorporation of chance state constraints (Chapter 6), was applied in simulation to the Buck-Boost converter. The states were still steered to a neighborhood of the desired target in the presence of chance constraints (input-to-state stability), and, through 300 disturbance realizations, it was shown that the minimal probability of constraint satisfaction was achieved.

[9] The domains of attraction were estimated by verifying feasibility of a grid of points via the barrier (*phase I*) method [36]

- NMPC for regulation without *offset* of the CSTR system: The NMPC with constant disturbance attenuation proposed in Chapter 3 was applied in simulation to the CSTR system. The process of estimating the disturbance mean-value and computing the disturbance propagation sets, taking into consideration the incorporation of a constant disturbance model in the prediction, were discussed. Regulation without steady-state *offset*, in the presence of disturbances with non-zero means, was achieved.

- NMPC for tracking of the CSTR system: The NMPC for tracking piece-wise constant references proposed in Chapter 5 was applied to the CSTR system. Recursive feasibility was observed and the controller steered the output to the neighborhood of the optimal admissible reference (input-to-state stability). The importance of the artificial reference was illustrated by showing the increase in the domain of attraction associated with it, and the effect of the *offset* cost weighting on the convergence of the artificial reference to its optimal value, and thus on the controller transient response, was analysed.

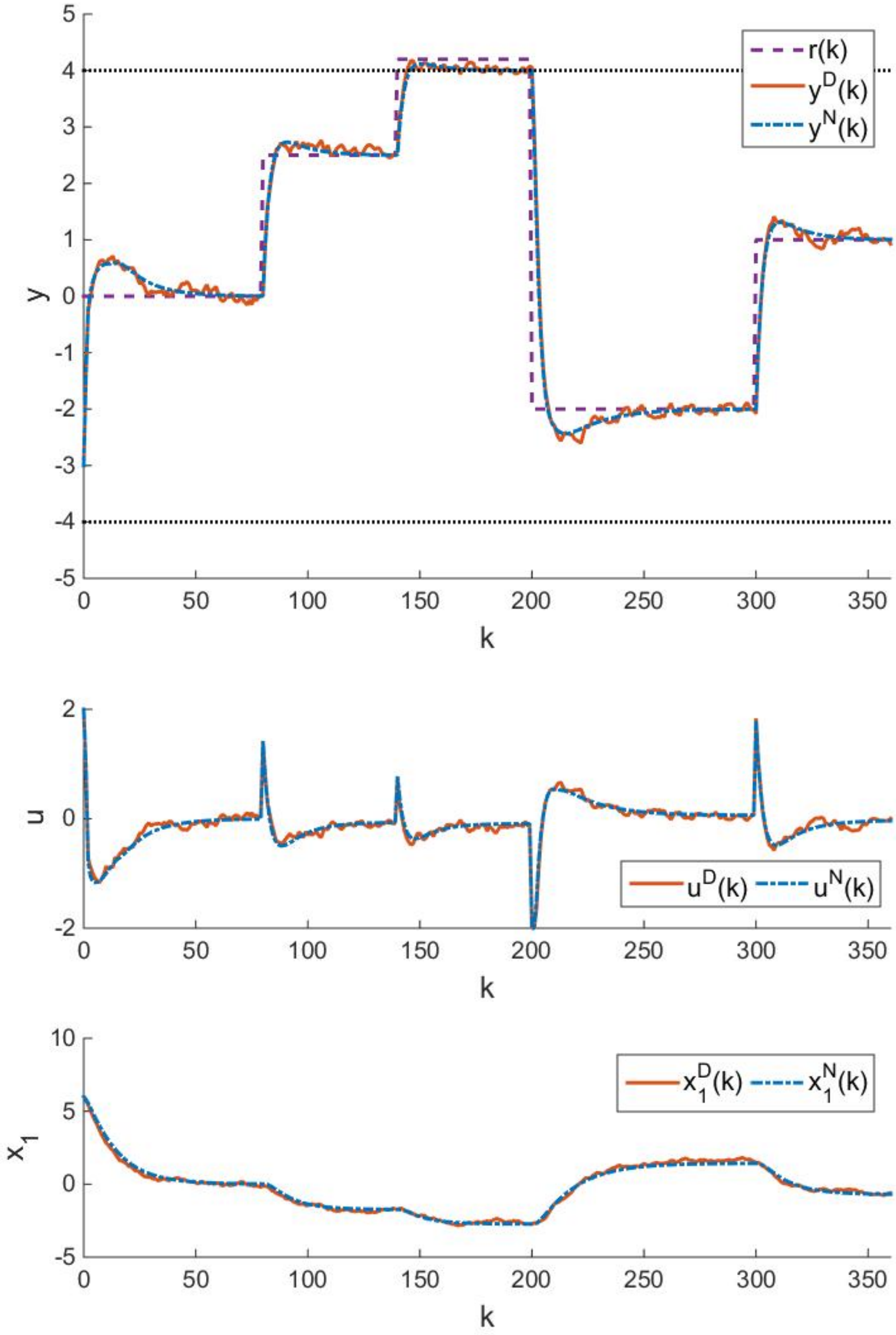


Figure 7.9: Simulation of the NMPC closed loop system subject to a piecewise constant reference ($r(k)$) for the nominal ($y^N(k)$) and disturbed ($y^D(k)$) cases. Top: output responses. Bottom: Input and first state responses.

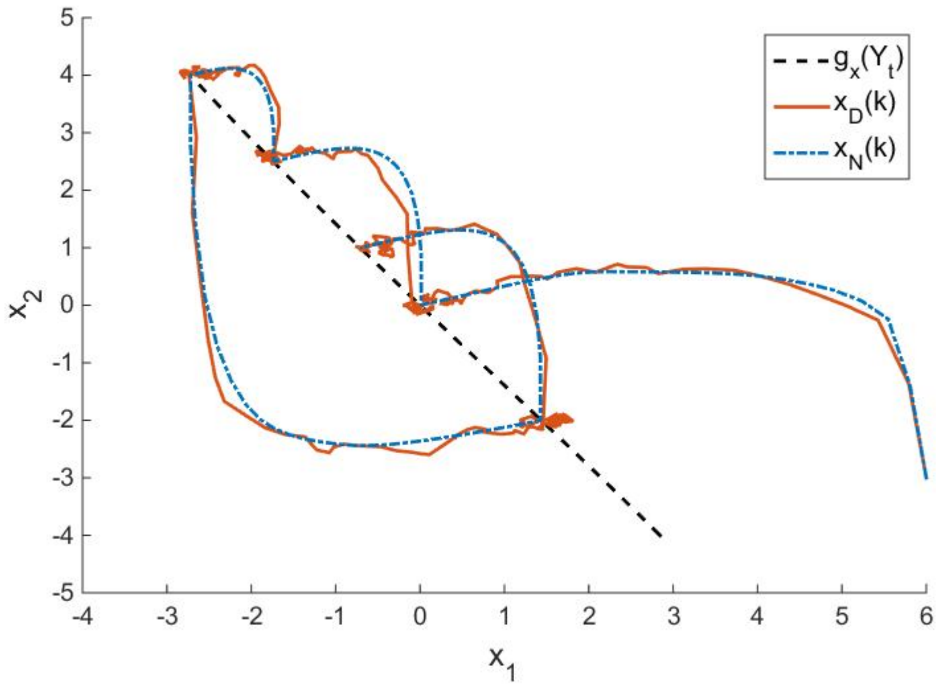
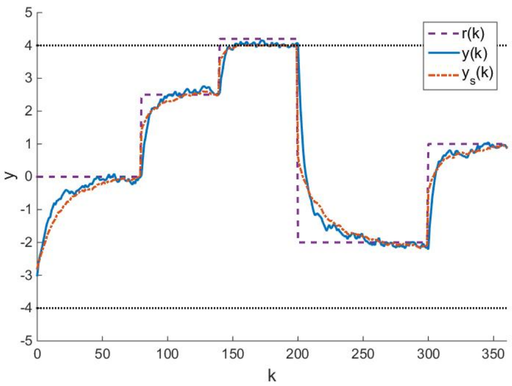
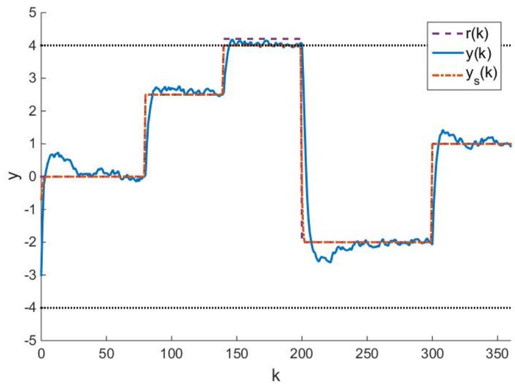


Figure 7.10: State-Space response of the NMPC closed loop system for the nominal ($x^N(k)$) and disturbed ($x^D(k)$) cases.



(a) $T = 20$.



(b) $T = 10^4$.

Figure 7.11: Comparison of the closed-loop responses for $T = 20$ and $T = 10^4$.

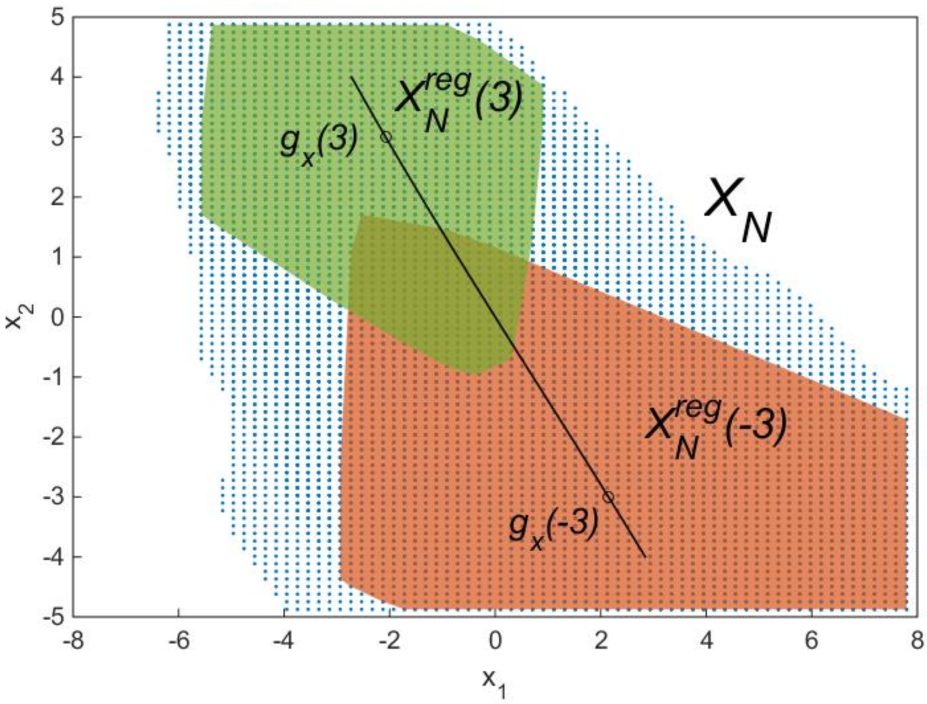


Figure 7.12: Domains of attraction for the tracking NMPC ($\mathcal{X}_N$) and regulation NMPC for $y_t = -3$ and $y_t = 3$ ($X_N^{reg}(-3)$ and $X_N^{reg}(3)$, respectively).

Chapter 8

Conclusion

This project presented robust Nonlinear Model Predictive Control (NMPC) strategies, using nominal predictions and constraint tightening in order to ensure recursive feasibility and robust constraint satisfaction. Furthermore, Input-to-State Stability (ISS) of the proposed controllers is guaranteed via ISS-Lyapunov analysis. The tightened constraints were based on zonotopic disturbance propagation sets, computed via mean-value zonotopic extension [1], which were shown to be less conservative than previous methods.

The robust NMPC for regulation [34, 19] was also adapted via the incorporation of constant disturbance estimations into the prediction model, in order to avoid offset in the presence of constant disturbances. The tracking NMPC problem was also tackled, extending the nominal result of [22] to the robust case. Furthermore, the incorporation of chance constraints [34] into the proposed control strategies, maintaining feasibility and stability guarantees, was considered, allowing a certain degree of admissible constraint violation to be specified in order to improve performance and augment the domain of attraction. Finally, the proposed controllers were applied to simulation Buck-Boost and CSTR (Continually Stirred Tank Reactor) case studies, in order to validate and compare the proposed approaches.

Further research could: (i) seek less conservative methods to compute Robust Positively Invariant (RPI) sets for nonlinear systems, (ii) study the extension of other feasibility guarantee methods, such as equality terminal constraints, to the robust case, (iii) combine the controllers of Chapters 3 and 5 for offset-free tracking, (iv) Consider the robust NMPC project without direct knowledge of all states, via output feedback or state estimation, and (v) Study the application of NMPC strategies for time-delay systems.

Bibliography

[1] T. Alamo, J.M. Bravo, and E.F. Camacho. "Guaranteed state estimation by zonotopes". In: *Automatica* 41.6 (2005), pp. 1035–1043. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2004.12.008.

[2] I. Alvarado. "Model Predictive control for tracking constrained linear systems". PhD thesis. Sevilla: Universidad de Sevilla, 2007.

[3] F. Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems*. Cambridge University Press, 2017. DOI: 10.1017/9781139061759.

[4] S. Boyd et al. *Linear Matrix Inequalities in System and Control Theory*. Boca Raton: Siam, 1994.

[5] L. Chisci, J.A. Rossiter, and G. Zappa. "Systems with persistent disturbances: predictive control with restricted constraints". In: *Automatica* 37.7 (2001), pp. 1019–1028. ISSN: 0005-1098. DOI: https://doi.org/10.1016/S0005-1098(01)00051-6.

[6] V.M. Cunha and T.L.M. Santos. "Controle Preditivo não-linear Robusto com Propagação de Incertezas via Zonotopos". In: *Anais da Sociedade Brasileira de Automática* 2.1 (2020). DOI: 10.48011/asba.v2i1.1682.

[7] V.M. Cunha and T.L.M. Santos. "Robust nonlinear model predictive control based on nominal predictions with piecewise constant references and bounded disturbances". In: *International Journal of Robust and Nonlinear Control* (2022). DOI: https://doi.org/10.1002/rnc.6004.

[8] V.M. Cunha and T.L.M. Santos. "Robust Nonlinear Model Predictive Control with Bounded Disturbances based on Zonotopic Constraint Tightening". In: *Journal of Control, Automation and Electrical Systems* special issue CBA 2020 (2021).

[9] A. Ferramosca et al. "Robust MPC for tracking zone regions based on nominal predictions". In: *Journal of Process Control* 22.10 (2012), pp. 1966–1974. ISSN: 0959-1524. DOI: https://doi.org/10.1016/j.jprocont.2012.08.013.

[10] V.T. Hieu Le et al. *Zonotopes - From Guaranteed State-estimation to Control*. London: ISTE Ltd, 2013.

[11] Z. Jiang and Y. Wang. “A converse Lyapunov theorem for discrete-time systems with disturbances”. In: *Systems & Control Letters* 45.1 (2002), pp. 49–58. ISSN: 0167-6911. DOI: https://doi.org/10.1016/S0167-6911(01)00164-5.

[12] Z. Jiang and Y. Wang. “Input-to-state stability for discrete-time nonlinear systems”. In: *Automatica* 37.6 (2001), pp. 857–869. ISSN: 0005-1098. DOI: https://doi.org/10.1016/S0005-1098(01)00028-0.

[13] J. Köhler, M. A. Müller, and F. Allgöwer. “A novel constraint tightening approach for nonlinear robust model predictive control”. In: *2018 Annual American Control Conference (ACC)*. 2018, pp. 728–734.

[14] I. Kolmanovsky and E.G. Gilbert. “Theory and computation of disturbance invariant sets for discrete-time linear systems”. In: *Mathematical Problems in Engineering* 4 (1998), pp. 317–367. DOI: https://doi.org/10.1155/S1024123X98000866.

[15] M.V. Kothare, V. Balakrishnan, and M. Morari. “Robust constrained model predictive control using linear matrix inequalities”. In: *Automatica* 32.10 (1996), pp. 1361–1379. ISSN: 0005-1098. DOI: https://doi.org/10.1016/0005-1098(96)00063-5.

[16] B. Kouvaritakis and M. Cannon. *Model Predictive Control*. New York: Springer International Publishing, 2016.

[17] M. Lazar et al. “Input-to-state stabilizing sub-optimal NMPC with an application to DC–DC converters”. In: *International Journal of Robust and Nonlinear Control* 18 (2008), pp. 890–904.

[18] E.L. Lima. *Curso de Análise vol.2 - 11ª edição*. Rio de Janeiro: Projeto Euclides - IMPA, 2014.

[19] D. Limon, T. Alamo, and E. F. Camacho. “Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties”. In: *Proceedings of the 41st IEEE Conference on Decision and Control, 2002*. Vol. 4. 2002, pp. 4619–4624.

[20] D. Limon et al. “Input-to-state stability: a unifying framework for robust model predictive control”. In: *Nonlinear model predictive control*. Springer, 2009, pp. 1–26.

[21] D. Limon et al. “MPC for tracking piecewise constant references for constrained linear systems”. In: *Automatica* 44.9 (2008), pp. 2382–2387. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2008.01.023.

[22] D. Limon et al. “Nonlinear MPC for Tracking Piece-Wise Constant Reference Signals”. In: *IEEE Transactions on Automatic Control* 63.11 (2018), pp. 3735–3750. DOI: 10.1109/TAC.2018.2798803.

[23] D. Limón. "Control predictivo de sistemas no lineales con restricciones: estabilidad y robustez". PhD thesis. Sevilla: Universidad de Sevilla, 2002.

[24] L. Magni et al. "A stabilizing model-based predictive control algorithm for nonlinear systems". In: *Automatica* 37.9 (2001), pp. 1351–1362.

[25] D.Q. Mayne. "Model predictive control: Recent developments and future promise". In: *Automatica* 50.12 (2014), pp. 2967–2986. ISSN: 0005-1098. DOI: `https://doi.org/10.1016/j.automatica.2014.10.128`.

[26] D.Q. Mayne, M.M. Seron, and S.V. Raković. "Robust model predictive control of constrained linear systems with bounded disturbances". In: *Automatica* 41.2 (2005), pp. 219–224. ISSN: 0005-1098. DOI: `https://doi.org/10.1016/j.automatica.2004.08.019`.

[27] D.Q. Mayne et al. "Constrained model predictive control: Stability and optimality". In: *Automatica* 36.6 (2000), pp. 789–814. ISSN: 0005-1098. DOI: `https://doi.org/10.1016/S0005-1098(99)00214-9`.

[28] R.E. Moore, R.B. Kearfott, and M.J. Cloud. *Introduction to Interval Analysis*. Philadelphia: Society of Industrial and Applied Mathematics, 2009.

[29] M.M. Morato et al. "Robust Nonlinear Predictive Control through qLPV embedding and Zonotope Uncertainty Propagation". In: *4th IFAC Workshop on Linear Parameter Varying Systems*. 2021.

[30] J.A. Paulson, T.L.M. Santos, and A. Mesbah. "Mixed stochastic-deterministic tube MPC for offset-free tracking in the presence of plant-model mismatch". In: *Journal of Process Control* 83 (2019), pp. 102–120. ISSN: 0959-1524. DOI: `https://doi.org/10.1016/j.jprocont.2018.04.010`.

[31] B.S. Rego et al. "Guaranteed methods based on constrained zonotopes for set-valued state estimation of nonlinear discrete-time systems". In: *Automatica* 111 (2020), p. 108614. ISSN: 0005-1098. DOI: `https://doi.org/10.1016/j.automatica.2019.108614`.

[32] J.A. Rossiter. *Model-Based Predictive Control: a practical approach*. Philadelphia: CRC Press, 2004.

[33] T.L.M. Santos and V.M. Cunha. "Robust MPC for linear systems with bounded disturbances based on admissible equilibria sets". In: *International Journal of Robust and Nonlinear Control* 31.7 (2021), pp. 2690–2711. DOI: `https://doi.org/10.1002/rnc.5409`.

[34] T.L.M. Santos et al. "A Constraint-Tightening Approach to Nonlinear Model Predictive Control with Chance Constraints for Stochastic Systems, In: 2019 Annual American Control Conference (ACC)". In: 2019, pp. 1641–1647.

[35] J.K. Scott et al. "Constrained zonotopes: A new tool for set-based estimation and fault detection". In: *Automatica* 69 (2016), pp. 126–136. ISSN: 0005-1098. DOI: `https://doi.org/10.1016/j.automatica.2016.02.036`.

[36] B. Stephen and L. Vanderberghe. *Convex Optimization*. New York: Cambridge University Press, 2004.

[37] Z. Wan and M.V. Kothare. "Efficient scheduled stabilizing model predictive control for constrained nonlinear systems". In: *International Journal of Robust and Nonlinear Control* 13.3-4 (2003), pp. 331–346. DOI: `https://doi.org/10.1002/rnc.821`.

Appendix A

Complementary Algorithms

A.1 Pontryagin Difference of Polytope and Zonotope

In this section, an algorithm for the computation of the Pontryagin difference of a polytope $\mathcal{P}$ and a zonotope Z ($\mathcal{P} \ominus Z$) is presented [2, Lemma 6.5]. This is useful, for example, for the constraint tightening process in the case of zonotopic disturbance propagation sets and polyhedral constraints.

First, consider a halfspace $\mathcal{H} = \{x \in \mathbb{R}^n \colon h^\intercal x \leq r_0\}$, where $h \in \mathbb{R}^n$, $r_0 \in \mathbb{R}$, and a set $Z \subseteq \mathbb{R}^n$. Notice that if we define $\gamma_0 \in \mathbb{R}$ as $\gamma_0 = \max_{z \in Z} h^\intercal z$, then the Pontryagin difference $\mathcal{H} \ominus Z$ is given by:

$$\mathcal{H} \ominus Z = \{x \in \mathbb{R}^n \colon h^\intercal x \leq r_0 - \gamma_0\}. \tag{A.1}$$

Considering $Z = c \oplus G\mathcal{B}_\infty^{n_g}$, γ_0 can be algebraically calculated as:

$$\gamma_0 = \max_{\xi \in \mathcal{B}_\infty^{n_g}} h^\intercal (c + G\xi) = h^\intercal c + \sum_{i=1}^{n_g} |a_i|, \tag{A.2}$$

where $a = h^\intercal G$. Therefore, given a polytope $\mathcal{P} = \{x \in \mathbb{R}^n \colon Hx \leq r\}$, $H \in \mathbb{R}^{m \times n}$, $r \in \mathbb{R}^m$, which can be seen as the intersection of halfspaces $\mathcal{H}_j = \{x \in \mathbb{R}^n \colon H_{j,:}\, x \leq r_j\}$, $j = 1 \ldots m$[1], and a zonotope $Z = c \oplus G\mathcal{B}_\infty^{n_g}$, the Pontryagin difference $\mathcal{P} \ominus Z$ is given by:

$$\mathcal{P} \ominus Z = \{x \in \mathbb{R}^n \colon Hx \leq r - \gamma\}, \tag{A.3}$$

where $\gamma \in \mathbb{R}^m$ is a vector defined by $\gamma_j = H_{j,:}\, c + \sum_{i=1}^{n_g} |A_{ij}|$, with $A = HG$. Notice that this operation results in another polytope, does not require any optimization and has a negligible computational cost.

[1] $H_{j,:}$ is the j-th row of H, i.e. $H_{j,:} = (h_{j1}, h_{j2}, \ldots, h_{jn})$.

A.2 Choice of Feedback Matrix K_v

It is presented here a procedure for choosing the feedback matrix K_v with the objective of reducing the disturbance propagation sets $\mathcal{S}_z(j)$[2]. Considering Theorem 4.3, a way of reducing the sets $\mathcal{S}_z(j)$ is by minimizing the Lipschitz constant $L_x = \max_{J\in\mathbf{J}_\pi} \|J\|_\infty$.

From (4.6), being $\mathbf{A} = \mathbf{J}_x$ e $\mathbf{B} = \mathbf{J}_u$ previously computed by means of interval arithmetic, we have

$$\begin{aligned} mid(\mathbf{J}_\pi) &= mid(\mathbf{A}) + mid(\mathbf{B})K_v = M_a + M_b K_v, \\ rad(\mathbf{J}_\pi) &= rad(\mathbf{A}) + rad(\mathbf{B})|K_v| = R_a + R_b|K_v|, \end{aligned} \tag{A.4}$$

and the Lipschitz constant L_x is given by

$$\begin{aligned} L_x &= \max_i \sum_{j=1}^{n} (|mid(\mathbf{J}_\pi)_{ij}| + rad(\mathbf{J}_\pi)_{ij}) \\ &= \max_i \sum_{j=1}^{n} (|(M_a + M_b K_v)_{ij}| + (R_a + R_b|K_v|)_{ij}). \end{aligned} \tag{A.5}$$

Therefore, the matrix K_v that minimizes L_x can be obtained from the solution of the following optimization problem:

$$\begin{aligned} &\min_{K_v, P, \gamma} \ \gamma \\ s.a: &\begin{cases} P \geq |M_a + M_b K_v| + (R_a + R_b|K_v|) \\ \gamma \geq \sum_{j=1}^{n} P_{ij}, \quad i = 1 \dots n \end{cases}, \end{aligned} \tag{A.6}$$

which can be converted into the linear program:

$$\begin{aligned} &\min_{K_v, \overline{K}_v, P, \gamma} \ \gamma \\ s.a: &\begin{cases} \overline{K}_v \geq K_v, \quad \overline{K}_v \geq -K_v \\ P \geq (M_a + M_b K_v) + (R_a + R_b \overline{K}_v) \\ P \geq -(M_a + M_b K_v) + (R_a + R_b \overline{K}_v) \\ \gamma \geq \sum_{j=1}^{n} P_{ij}, \quad i = 1 \dots n \end{cases}, \end{aligned} \tag{A.7}$$

with $n^2 + 2nm + 1$ variables and $2n^2 + 2nm + n$ restrictions.

A.3 Zonotopic Order Reduction

In this section, an algorithm for reducing the number of generators of a zonotope, proposed in [31], is presented. First, notice that only a method to reduce the number of

[2]For simplicity of notation, $\mathcal{S}_z(j)$ is used here to represent zonotopic disturbance propagation sets obtained from any of the proposed methods (Properties 4.1, 4.2 or 4.3).

generators by 1 is needed, with further reductions being possible by recursive applications of this algorithm. We can also consider, without loss of generality, centered zonotopes.

Therefore, given a zonotope $Z = G\mathcal{B}_\infty^{n_g} \subseteq \mathbb{R}^n$, we want to find a zonotope $\overline{Z} = \overline{G}\mathcal{B}_\infty^{n_g-1}$ (ideally the smallest possible) such that $Z \subseteq \overline{Z}$. The first step consists on reducing G via *Gauss-Jordan* elimination with full pivoting to row echelon form[3] $\begin{pmatrix} I & R \end{pmatrix}$, where $|r_{ij}| \leq 1,\ \forall i, j$ can be ensured by choosing as pivot in each iteration of the elimination the element of the unreduced submatrix with largest module relative to the norm of the row it occupies.[4]

We can thus reorder G as $\begin{pmatrix} T & V \end{pmatrix}$, where $T \in \mathbb{R}^{n \times n}$ is nonsingular and $R = T^{-1}V$. A column $V_{:\,,j}$ is then chosen from V, with Z being represented by

$$Z = X \oplus Y = \begin{pmatrix} T & V_{:\,,j} \end{pmatrix} \mathcal{B}_\infty^{n+1} \oplus V_{-}\mathcal{B}_\infty^{n_g-n-1}, \tag{A.8}$$

where V_{-} represents the matrix formed by removing $V_{:\,,j}$ from V. X is then, conservatively, reduced to the parallelotope $\overline{X} = T\left(I + diag|R_{:\,,j}|\right)$, with $\overline{Z} = \overline{X} \oplus Y$.

Finally, the row $V_{:\,,j}$, which will be incorporated into T, can be chosen so as to minimize the increase in volume $v(\overline{X}) - v(X)$. This is equivalent to finding the $j \in \mathbb{Z}_{[n+1,n_g]}$ that minimizes

$$\Delta(j) = \prod_{i=1}^{n}(1 + |r_{ij}|) - \left(1 + \sum_{i=1}^{n} |r_{ij}|\right). \tag{A.9}$$

Remark A.1. *For eliminating k generators recursively, it is possible to make the* Gauss-Jordan *elimination only once, with the matrices T and R being directly updated at each iteration by*

$$T^+ = T(I + diag|R_{:\,,j}|), \qquad R^+ = (I + diag|R_{:\,,j}|)^{-1}R. \tag{A.10}$$

[3]Notice that if G is not full line rank, Z is degenerated, having less than n dimensions. Therefore, this reduction is always possible if Z has a non-empty interior, which is generally the case for all zonotopes considered in this work.

[4]Given a matrix $A \in \mathbb{R}^{m \times n}$, the norm of a row $A_{i,:}$ is considered here as the induced infinity-norm of the linear transformation $A_{i,:} : \mathbb{R}^n \to \mathbb{R}$, i.e. $\|A_{i,:}\|_\infty = \sum_{j=1}^{n} |a_{ij}|$.

Appendix B

Auxiliary Lemmas

Lemma B.1 (Input-to-State Lyapunov Stability [12]). *Consider system* (2.1) *subject to a control law* $u_k = \kappa(x_k)$. *Or, alternatively, the closed-loop system* $x_{k+1} = f_\kappa(x_k) + w_k = f(x_k, \kappa(x_k)) + w_k$, *where* $w_k \in \mathcal{W}$ *is viewed as an input. Given* $\mathcal{X} \subseteq \mathbb{R}^n$ *a robust positively invariant set of the closed-loop system*[1], *assume that there exists a function* $W\colon \mathbb{R}^n \to \mathbb{R}$ *such that, for all* $x_k \in \mathcal{X}$*:*

$$\alpha_1(\|x_k - \overline{x}\|) \leq W(x_k) \leq \alpha_2(\|x_k - \overline{x}\|), \tag{B.1a}$$

$$W(x_{k+1}) - W(x_k) \leq -\alpha_3(\|x_k - \overline{x}\|) + \theta(\left\|\mathbf{w}_{[0,k]}\right\|), \tag{B.1b}$$

where $\overline{x} \in \mathcal{X}$ *is a given constant state,* α_1, α_2 *and* α_3 *are* $\mathcal{K}_\infty$*-functions, and* θ *is a* $\mathcal{K}$*-function. Then,* $W(x_k)$ *is said to be an Input-to-State Lyapunov function and the closed-loop system is Input-to-State Stable (ISS), i.e. there exist a* $\mathcal{KL}$*-function* β *and a* $\mathcal{K}$*-function* γ *such that, given any initial state* $x_0 \in \mathcal{X}$*:*

$$\|x_k - \overline{x}\| \leq \beta(\|x_0 - \overline{x}\|, k) + \gamma(\left\|\mathbf{w}_{[0,k]}\right\|), \quad \forall k \in \mathbb{N}. \tag{B.2}$$

Proof. The proof arguments are the same of related works [12, 20]. However, a brief demonstration is presented here to provide a self-contained stability analysis. In the following, $\circ$ is used to represent function composition and $\mathrm{id}\colon \mathbb{R}^+ \to \mathbb{R}^+$ to represent the identity function. Defining $\alpha_4 = \alpha_3 \circ \alpha_2^{-1}$ and $w_m = sup\{\|w_k\|,\ k \in \mathbb{N}\}$, Eq. (B.1b) can be rewritten as follows:

$$W(x_{k+1}) - W(x_k) \leq -\alpha_4(W(x_k)) + \theta(w_m). \tag{B.3}$$

Without loss of generality, assume that $\mathrm{id} - \alpha_4$ is a $\mathcal{K}$-function [2]. Let ρ be a $\mathcal{K}_\infty$-function such that $\mathrm{id} - \rho$ is also a $\mathcal{K}_\infty$-function, and consider the set $\mathcal{D} = \{x \in \mathbb{R}^n \colon W(x) \leq c\}$,

[1] i.e. $x_k \in \mathcal{X} \Rightarrow x_{k+1} = f_\kappa(x_k) + w_k \in \mathcal{X}$, for any $w_k \in \mathcal{W}$.

[2] This can be achieved replacing α_2 by $\tilde{\alpha}_2 = \alpha_3 + \tilde{\rho}$, where $\tilde{\rho}$ is a suitable $\mathcal{K}$-function such that $\tilde{\alpha}_2(s) \geq \alpha_2(s),\ \forall s \geq 0$.

where $c = \alpha_4^{-1} \circ \rho^{-1} \circ \theta(w_m)$. Now, the proof is divided in two steps: (i) $x_k \in \mathcal{D}$, and (ii) $x_k \notin \mathcal{D}$. For the case (i), $x_k \in \mathcal{D} \Rightarrow W(x_k) \leq c = \alpha_4^{-1} \circ \rho^{-1} \circ \theta(w_m)$, $\rho \circ \alpha_4(c) = \theta(w_m)$ by definition, and $(\mathrm{id} - \rho) \circ \alpha_4$ is a $\mathcal{K}$-function, being the composition of $\mathcal{K}$-functions. Then, from (B.3):

$$\begin{aligned} W(x_{k+1}) &\leq (\mathrm{id} - \alpha_4)(W(x_k)) + \theta(w_m) + \rho \circ \alpha_4(c) - \rho \circ \alpha_4(c) \\ &\leq (\mathrm{id} - \alpha_4)(c) + \rho \circ \alpha_4(c) + \theta(w_m) - \rho \circ \alpha_4(c) \\ &= (\rho \circ \alpha_4 - \alpha_4)(c) + c + \theta(w_m) - \rho \circ \alpha_4(c) \\ &= -(\mathrm{id} - \rho) \circ \alpha_4(c) + c \\ &\leq c. \end{aligned} \tag{B.4}$$

Therefore, the set $\mathcal{D}$ is robust positively invariant, as $x_k \in \mathcal{D} \Rightarrow x_{k+1} \in \mathcal{D}$, and if there exists $k_0 \in \mathbb{R}$ such that $x_{k_0} \in \mathcal{D}$, then $x_k \in \mathcal{D}$ and $W(x_k) \leq \alpha_4^{-1} \circ \rho^{-1} \circ \theta(w_m) = \hat{\gamma}(w_m)$ for all $k \geq k_0$, where $\hat{\gamma} = \alpha_4^{-1} \circ \rho^{-1} \circ \theta$ is a $\mathcal{K}$-function.

Now assume that $x_k \notin \mathcal{D}$, such that $V(x_k) > c$. Then, from the positive invariance of $\mathcal{D}$, $x_j \notin \mathcal{D}$, $\forall j \leq k$ and $\rho \circ \alpha_4(W(x_k)) > \theta(w_m)$. Then, from inequality (B.1b):

$$\begin{aligned} W(x_{k+1}) - W(x_k) &\leq -\alpha_4(W(x_k)) + \theta(w_m) + \rho \circ \alpha_4(W(x_k))) - \rho_f \circ \alpha_4(W(x_k)) \\ &= -(\mathrm{id} - \rho) \circ \alpha_4(W(x_k)) + \theta(w_m) - \rho \circ \alpha_4(W(x_k)) \\ &\leq -(\mathrm{id} - \rho) \circ \alpha_4(W(x_k)) \end{aligned} \tag{B.5}$$

and $W(x_k)$ decreases at each sampling instant by at least a $\mathcal{K}$-function of itself. Therefore, by a standard comparison lemma [11], there exists a $\mathcal{KL}$-function $\hat{\beta}$ such that $W(x_k) \leq \hat{\beta}(W(x_0), k)$ for all k which satisfies $x_k \notin \mathcal{D}$.

The two cases can then be combined, with $W(x_k) \leq \max\{\hat{\gamma}(w_m), \hat{\beta}(W(x_0), k)\} \leq \hat{\beta}(W(x_0), k) + \hat{\gamma}(w_m)$, for all $k \in \mathbb{N}$, and from (B.1a):

$$\begin{aligned} \|x_k - \overline{x}\| &\leq \alpha_1^{-1}(\hat{\beta}(\alpha_2(\|x_k - \overline{x}\|), k) + \hat{\gamma}(w_m)) \\ &= \beta(\|x_k - \overline{x}\|, k) + \gamma(w_m), \end{aligned} \tag{B.6}$$

where $\beta(s,t) = \alpha_1^{-1}(\hat{\beta}(\alpha_2(s), t))$ is a $\mathcal{KL}$-function and $\gamma = \alpha_1^{-1} \circ \hat{\gamma}$ is a $\mathcal{K}$-function. Due to the causality of the nonlinear system, $\gamma(w_m)$ can be replaced by $\gamma(\|\mathbf{w}_{[0,k]}\|)$, which completes the proof. □

Lemma B.2 ($\mathcal{K}$-function upper bound extension)**.** *Consider a couple of sets $\mathcal{X}, \Omega$ with $\mathcal{X} \subseteq \mathbb{R}^n$, $\Omega \subseteq \mathcal{X}$, and x^* in the interior of Ω. Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a scalar function such that there exists a finite constant $M > 0 \in \mathbb{R}$ such that $V(x) \leq M$ for all $x \in \mathcal{X}$ and there exists a $\mathcal{K}$-function α such that $V(x) \leq \alpha(\|x - x^*\|)$ for all $x \in \Omega$.*

Then, there exists a $\mathcal{K}_\infty$-function β such that $V(x) \leq \beta(\|x - x^*\|)$ for all $x \in \mathcal{X}$. Furthermore, if $\alpha(\|x - x^*\|) = b\|x - x^*\|^a$, with $b > 0, a > 1 \in \mathbb{R}$, then there exists $\bar{b} > 0 \in \mathbb{R}$ such that $V(x) \leq \bar{b}\|x - x^*\|^a$, $\forall x \in \mathcal{X}$.

Proof. Since x^* is in the interior of Ω, there exists $\varepsilon > 0 \in \mathbb{R}$ such that $\|x - x^*\| \geq \varepsilon,\ \forall x \notin \Omega$. Therefore, for any $x \in \mathcal{X} \setminus \Omega$, $a > 1 \in \mathbb{R}$, we have

$$\frac{V(x)}{\|x - x^*\|^a} \leq \bar{c}\,, \tag{B.7}$$

where $\mathcal{X} \setminus \Omega = \{x \in \mathcal{X} \colon x \notin \Omega\}$ and $\bar{c} = \varepsilon^{-a} M$. Finally, defining $\beta(\|x - x^*\|) = \max(\alpha(\|x - x^*\|), \bar{c}\,\|x - x^*\|^a)$ a $\mathcal{K}_\infty$-function, we have $V(x) \leq \beta(\|x - x^*\|),\ \forall x \in \mathcal{X}$. If $\alpha(\|x - x^*\|) = b\,\|x - x^*\|^a$, taking $\bar{b} = \max(b, \bar{c})$ results in $V(x) \leq \bar{b}\,\|x - x^*\|^a,\ \forall x \in \mathcal{X}$. This result is inspired by a related lemma with respect to the origin [20, Lemma 4]. □

Lemma B.3 (Quadratic offset cost)**.** *Consider the quadratic positive definite function $V_O \colon \mathcal{Y}_t \to \mathbb{R}$ given by $V_O(y) = y^\intercal T y$, $T \succ 0 \in \mathbb{R}^{p \times p}$, where $\mathcal{Y}_t \subseteq \mathbb{R}^p$ is a convex set. Given a $y_t \in \mathbb{R}^p$, let y_s^o be the minimizer of $V_O(y_s - y_t)$ for $y_s \in \mathcal{Y}_t$. Then, we have*

$$V_O(y_s - y_t) - V_O(y_s^o - y_t) \geq (y_s - y_s^o)^\intercal T (y_s - y_s^o), \quad \forall y_s \in \mathcal{Y}_t. \tag{B.8}$$

In particular, since T is positive definite, Eq. (5.10) is satisfied with $\alpha_O(\|y_s - y_s^o\|) = \lambda_{m,T}\,\|y_s - y_s^o\|^2$, where $\lambda_{m,T} > 0$ is the smallest eigenvalue of T.

Proof. First, notice that, from the convexity of $\mathcal{Y}_t$ and optimality of y_s^o, we have

$$\nabla^\intercal V_O(y_s - y_t)\big|_{y_s = y_s^o}(y_s - y_s^o) \geq 0, \quad \forall y_s \in \mathcal{Y}_t. \tag{B.9}$$

Since $\nabla V_O(y_s - y_t) = 2T(y_s - y_t)$, then $(y_s^o - y_t)^\intercal T (y_s - y_s^o) \geq 0$, and we have

$$\begin{aligned} V_O(y_s - y_t) &= (y_s - y_t)^\intercal T (y_s - y_t) = (y_s^0 - y_t + (y_s - y_s^o))^\intercal T (y_s^0 - y_t + (y_s - y_s^o)) \\ &= (y_s^o - y_t)^\intercal T (y_s^o - y_t) + 2(y_s^o - y_t)^\intercal T (y_s - y_s^o) + (y_s - y_s^o)^\intercal T (y_s - y_s^o) \\ &\geq V_O(y_s^o - y_t) + (y_s - y_s^o)^\intercal T (y_s - y_s^o). \end{aligned} \tag{B.10}$$

□

Lemma B.4. *Consider the tracking NMPC optimal control problem $P_N^t(x_k, y_t)$ and let Assumptions 5.1, 5.2, 5.3 and 5.4 hold. Then, for any feasible reference $y_s \in \mathcal{Y}_t$ and target $y_t \in \mathbb{R}^p$, we have:*

$$\|x_k - x_s\| < \varepsilon_s \quad \Rightarrow \quad V_N^*(x_k, y_t) \leq V_f(x_k - x_s, y_s) + V_O(y_s - y_t), \tag{B.11}$$

where $x_s = \varrho_x(y_s)$ and ε_s is given as in Assumption 5.4 (ii).

Proof. First, consider a cut of the terminal set Γ for $y_s \in \mathcal{Y}_t$, given by: $\Gamma_{y_s} = \{x \in \mathbb{R}^n \colon (x, y_s) \in \Gamma\}$. The admissibility of $\Gamma \subseteq \Lambda_N$ ensures that $(x, v_t(x, y_s)) \in \mathcal{Z}_\pi(N) \subseteq \mathcal{Z}_\pi(j),\ \forall j \in [0, N]$, for any $x \in \Gamma_{y_s}$ and, from the RPI definition of Γ, $x_0 \in \Gamma_{y_s}$ with $x_{j+1} = f_\pi(x_j, v_t(x_j, y_s)),\ \forall j \in [0, N-1]$, is such that $x_j \in \Gamma_{y_s},\ \forall j \in [0, N]$.

In other words, the unconstrained terminal control law obtained from y_s is a feasible candidate for $P_N^t(x_k, y_t)$ if $x_k \in \Gamma_{y_s}$. Therefore, from the decreasing terminal cost assumption (5.11c) and optimality of the effective solution:

$$V_N^*(x_k, y_t) \leq V_N^{y_s}(x_k, y_t) \leq V_f(x_k - x_s, y_s) + V_O(y_s - y_t), \tag{B.12}$$

for all $x_k \in \Gamma_{y_s}$, where $V_N^{y_s}(x_k, y_t)$ represents the candidate solution obtained from the terminal control law with artificial reference y_s.

Finally, from the RPI definition of Γ, since $x_s \in \Gamma_{y_s}$, then $f(x_s, u_t(x_s, y_s)) \oplus \mathcal{S}(N) = x_s \oplus \mathcal{S}(N) \subseteq \Gamma_{y_s}$ and, since the origin is an interior point of $\mathcal{S}(N)$, $\|x_k - x_s\| < \varepsilon_s \Rightarrow x_k - x_s \in \mathcal{S}(N) \Rightarrow x_k \in \Gamma_{y_s}$, which completes the proof. $\square$

Lemma B.5. *Consider the NMPC optimization problem $P_N^t(x_k, y_t)$, $x_k \in \mathcal{X}_N$, $y_t \in \mathbb{R}^p$. Let $x_{s,k}^* = \varrho_x(y_{s,k}^*)$, where $y_{s,k}^* = y_s^*(x_k, y_t)$ is the artificial reference associated to the solution of $P_N(x_k, y_t)$, and $x_s^o = \varrho_x(y_s^o)$, where y_s^o is given as in* (5.9)*. If Assumptions 5.1, 5.2, 5.3 and 5.4 are satisfied, then there exists a $\mathcal{K}_\infty$-function α_V such that:*

$$\|x_k - x_s^o\| \leq \alpha_V(\|x_k - x_{s,k}^*\|), \tag{B.13}$$

for all $x_k \in \mathcal{X}_N$ and $y_t \in \mathbb{R}^p$.

Proof. For simplicity of notation, let $\varepsilon_x = \varepsilon_s/2$, where ε_s is given as in Assumption 5.4 (ii), $W^*(x_k, y_t) = V_N^*(x_k, y_t) - V_O(y_s^o - y_t)$ and $L_g \in \mathbb{R}$ be a Lipschitz constant for ϱ_x in $\mathcal{Y}_t$, i.e. $\|\varrho_x(y_b) - \varrho_x(y_a)\| \leq L_g \|y_b - y_a\|$, $\forall y_a, y_b \in \mathcal{Y}_t$. Now, two cases are considered: (i) $\|x_k - x_{s,k}^*\| < \varepsilon_x$ and (ii) $\|x_k - x_{s,k}^*\| \geq \varepsilon_x$.

Assuming $\|x_k - x_{s,k}^*\| < \varepsilon_x$, consider $y_\eta = \lambda_c y_{s,k}^* + (1 - \lambda_c) y_s^o$, with $0 < \lambda_c < 1$ and $x_\eta = \varrho_x(y_\eta)$ [3]. From the compactness of $\mathcal{Y}_t$, there exists $M_y \in \mathbb{R}$ such that $\|y_{s,k}^* - y_s^o\| \leq M_y$, $\forall y_{s,k}^*, y_s^o \in \mathcal{Y}_t$. Consider $\tilde{\lambda}_c$ sufficiently close to 1 such that $(1 - \tilde{\lambda}_c) < \epsilon_x (L_g M_y)^{-1}$. Then, $\|x_\eta - x_{s,k}^*\| \leq L_g \|y_\eta - y_{s,k}^*\| = L_g (1 - \lambda_c) \|y_{s,k}^* - y_s^o\| < \epsilon_x$ and $\|x_k - x_\eta\| \leq \|x_k - x_{s,k}^*\| + \|x_{s,k}^* - x_\eta\| < 2\epsilon_x = \varepsilon_s$ for all $\lambda_c \geq \tilde{\lambda}_c$.

Therefore, for $\lambda_c \geq \tilde{\lambda}_c$ we have $\|x_k - x_\eta\| < \varepsilon_s$ and, from Lemma B.4, the following upper bound for $W^*(x_k, y_t)$ can be defined:

$$\begin{aligned}
W^*(x_k, y_t) &\leq V_f(x_k - x_\eta) + V_O(y_\eta - y_t) - V_O(y_s^o - y_t) \\
&\leq b \|x_k - x_\eta\|^a + V_O(y_\eta - y_t) - \lambda_c V_O(y_s^o - y_t) - (1 - \lambda_c) V_O(y_s^o - y_t) \\
&\leq b \|x_k - x_{s,k}^* + x_{s,k}^* - x_\eta\|^a + \lambda_c (V_O(y_{s,k}^* - y_t) - V_O(y_s^o - y_t)) \\
&\qquad - (1 - \lambda_c)(V_O(y_s^o - y_t) - V_O(y_s^o - y_t)) \\
&\leq 2^a b \|x_k - x_{s,k}^*\|^a + 2^a b L_g^a \|y_{s,k}^* - y_\eta\|^a + \lambda_c (V_O(y_{s,k}^* - y_t) - V_O(y_s^o - y_t)) \\
&= 2^a b \|x_k - x_{s,k}^*\|^a + 2^a b L_g^a (1 - \lambda_c)^a \|y_{s,k}^* - y_s^o\|^a \\
&\qquad + \lambda_c (V_O(y_{s,k}^* - y_t) - V_O(y_s^o - y_t)),
\end{aligned} \tag{B.14}$$

[3] Notice that, from the convexity of $\mathcal{Y}_t$, $y_\eta \in \mathcal{Y}_t$.

where the convexity of $V_O(\cdot)$, $\|p_1+p_2\|^a \leq \max(2^a\|p_1\|^a, 2^a\|p_2\|^a)$ and $y_\eta - y^*_{s,k} = (1-\lambda_c)(y^o_s - y^*_{s,k})$ were used. Furthermore, from $W^*(x_k, y_t) \geq V_O(y^*_{s,k} - y_t) - V_O(y^o_s - y_t)$, we have

$$\begin{aligned} V_O(y^*_{s,k} - y_t) - V_O(y^o_s - y_t) \leq\; & 2^a b \left\|x_k - x^*_{s,k}\right\|^a + 2^a b L_g^a (1-\lambda_c)^a \left\|y^*_{s,k} - y^o_s\right\|^a \\ & + \lambda_c (V_O(y^*_{s,k} - y_t) - V_O(y^o_s - y_t)), \end{aligned} \tag{B.15}$$

or alternatively,

$$V_O(y^*_{s,k} - y_t) - V_O(y^o_s - y_t) \leq (1-\lambda_c)^{-1} 2^a b \left\|x_k - x^*_{s,k}\right\|^a + 2^a b L_g^a (1-\lambda_c)^{a-1} \left\|y^*_{s,k} - y^o_s\right\|^a. \tag{B.16}$$

Now, since $V_O(y^*_{s,k} - y_t) - V_O(y^o_s - y_t) \geq \alpha_O(\left\|y^*_{s,k} - y_t\right\|)$,

$$\alpha_O(\left\|y^*_{s,k} - y_t\right\|) \leq (1-\lambda_c)^{-1} 2^a b \left\|x_k - x^*_{s,k}\right\|^a + 2^a b L_g^a (1-\lambda_c)^{a-1} \left\|y^*_{s,k} - y^o_s\right\|^a \tag{B.17}$$

and, for any $0 < c_\alpha < 1 \in \mathbb{R}$,

$$\begin{aligned} c_\alpha \alpha_O(\left\|y^*_{s,k} - y^o_s\right\|) \leq & -\left((1-c_\alpha)\alpha_O(\left\|y^*_{s,k} - y^o_s\right\|) - 2^a b L_g^a (1-\lambda_c)^{a-1} \left\|y^*_{s,k} - y^o_s\right\|^a\right) \\ & + (1-\lambda_c)^{-1} 2^a b \left\|x_k - x^*_{s,k}\right\|^a. \end{aligned} \tag{B.18}$$

Based on Assumption 5.4(i) and defining $\bar{c}_s = \min(c_s, M_y^{-a}\alpha_O(s_0))$, we have $\alpha_O(\left\|y^*_{s,k} - y^o_s\right\|) \geq \bar{c}_s \left\|y^*_{s,k} - y^o_s\right\|^a$, $\forall y^*_{s,k}, y^o_s \in \mathcal{Y}_t$. Therefore, since $a > 1$ and $\lim_{\lambda_c \to 1}(1-\lambda_c)^{a-1} = 0$, there exists a $\overline{\lambda_c} \geq \tilde{\lambda}_c$ such that $(1-\overline{\lambda_c})^{a-1} \leq (2^a b L_g^a)^{-1}(1-c_\alpha)\bar{c}_s$. Then $(1-c_\alpha)\alpha_O(\left\|y^*_{s,k} - y^o_s\right\|) \geq 2^a b L_g^a (1-\overline{\lambda_c})^{a-1} \left\|y^*_{s,k} - y^o_s\right\|^a$ and thus

$$\begin{aligned} c_\alpha \alpha_O(\left\|y^*_{s,k} - y^o_s\right\|) &\leq (1-\overline{\lambda_c})^{-1} 2^a b \left\|x_k - x^*_{s,k}\right\|^a, \\ \left\|y^*_{s,k} - y^o_s\right\| &\leq \alpha_O^{-1}\left(c_\alpha^{-1}(1-\overline{\lambda_c})^{-1} 2^a b \left\|x_k - x^*_{s,k}\right\|^a\right), \\ \left\|x^*_{s,k} - x^o_s\right\| &\leq L_g\ \alpha_O^{-1}\left(c_\alpha^{-1}(1-\overline{\lambda_c})^{-1} 2^a b \left\|x_k - x^*_{s,k}\right\|^a\right). \end{aligned} \tag{B.19}$$

Finally, defining the $\mathcal{K}$-function $\tilde{\alpha}_V(s) = s + L_g \alpha_O^{-1}\left(c_\alpha^{-1}(1-\overline{\lambda_c})^{-1} 2^a b\, s^a\right)$ and using the triangular inequality, we have

$$\|x_k - x^o_s\| \leq \left\|x_k - x^*_{s,k}\right\| + \left\|x^*_{s,k} - x^o_s\right\| \leq \tilde{\alpha}_V(\left\|x_k - x^*_{s,k}\right\|). \tag{B.20}$$

Inequality (B.13) is thus satisfied for $\left\|x_k - x^*_{s,k}\right\| < \varepsilon_x$. For the case $\left\|x_k - x^*_{s,k}\right\| \geq \varepsilon_x$, a similar argument of Lemma B.2 can be used. From the compactness of the constraints, there exists $M_x \in \mathbb{R}$ such that $\|x_k - x^o_s\| \leq M_x$, $\forall x_k \in \mathcal{X}_N, y^o_s \in \mathcal{Y}_t$. Defining $\tilde{c}_x = \varepsilon_x^{-1} M_x$ and $\alpha_V(s) = \max(\tilde{\alpha}_V(s), \tilde{c}_x s)$, then

$$\|x_k - x^o_s\| \leq \alpha_V(\left\|x_k - x^*_{s,k}\right\|) \tag{B.21}$$

for both cases, where α_V is a $\mathcal{K}_\infty$-function. □

Lemma B.6. *Consider the optimization problems $\tilde{P}_N(x_k)$, $\tilde{P}^{\mu}_N(x_k, \hat{\mu}_k)$ and $\tilde{P}^t_N(x_k, y_t)$ defined in Chapter 6, under the additional Assumption 6.1. The unconstrained terminal control law is always admissible inside the terminal set. That is:*

(i) In the regulation case ($\tilde{P}_N(x_k)$ and $\tilde{P}^{\mu}_N(x_k, \hat{\mu}_k)$), the control sequence recursively defined by $v_{k+j|k} = v_t(x_{k+j|k})$, $j = 1 \ldots N-1$ is admissible if $x_k \in \mathcal{X}_f$.

(ii) In the tracking case ($\tilde{P}^t_N(x_k, y_t)$), the artificial variable y_s and control sequence $v_{k+j|k} = v_t(x_{k+j|k}, y_s)$, $j = 1 \ldots N-1$ is admissible if $(x_k, y_s) \in \Gamma$.

Proof. The proof is divided into the two cases: regulation and tracking, respectively.

(i) From the admissibility of the terminal set ($\mathcal{X}_f \subseteq \mathcal{V}_N$), we have $(x, v_t(x)) \in \mathcal{Z}_\pi(N) \subseteq \mathcal{Z}_\pi(j)$, $\forall j \in [0, N]$ and $x \in \mathcal{X}^{cc}(N) \subseteq \mathcal{X}^{cc}(j)$, $\forall j \in [1, N]$, for any $x \in \mathcal{X}_f$. If $x_k \in \mathcal{X}_f$, from the positive invariance of $\mathcal{X}_f$ we have $x_{k+j|k} \in \mathcal{X}_f$, $j = 0 \ldots N$ if the terminal control law is considered. Therefore, the constraints $(x_{k+j|k}, v_t(x_{k+j|k})) \in \mathcal{Z}_\pi(j)$, $j = 0 \ldots N-1$, $x_{k+j|k} \in \mathcal{X}^{cc}(j)$, $j = 1 \ldots N-1$ and $x_{k+N|k} \in \mathcal{X}_f$ are satisfied and the terminal control law is feasible in the entire prediction horizon.

(ii) From the admissibility of the terminal set ($\Gamma \subseteq \Lambda_N$), we have $(x, v_t(x, y_s)) \in \mathcal{Z}_\pi(N) \subseteq \mathcal{Z}_\pi(j)$, $\forall j \in [0, N]$ and $x \in \mathcal{X}^{cc}(N) \subseteq \mathcal{X}^{cc}(j)$, $\forall j \in [1, N]$, for any $(x, y_s) \in \Gamma$. If $(x_k, y_s) \in \Gamma$, being Γ a positive invariant set for tracking, $(x_{k+j|k}, y_s) \in \Gamma$, $j = 0 \ldots N$ if the terminal control law with artificial reference y_s is considered. Therefore, the constraints $(x_{k+j|k}, v_t(x_{k+j|k}, y_s)) \in \mathcal{Z}_\pi(j)$, $j = 0 \ldots N-1$, $x_{k+j|k} \in \mathcal{X}^{cc}(j)$, $j = 1 \ldots N-1$ and $(x_{k+N|k}, y_s) \in \Gamma$ are satisfied and the terminal control law, with artificial reference y_s, is feasible.

$\square$

Appendix C

Zonotopic Inclusion Properties

In this appendix, additional properties of the zonotopic inclusion (Theorem 4.1) are deduced. First, consider the linear operator $\iota(\cdot)$ which takes a matrix $A \in \mathbb{R}^{n\times m}$ into the diagonal matrix $P \in \mathbb{R}^{n\times n}$ whose elements are the sums of the lines of A, i.e. $\iota(A) = (p_{ij})_{n\times n}$, with

$$p_{ii} = \sum_{j=1}^{m} a_{ij},\ i = 1 \dots n, \qquad p_{ij} = 0,\ \forall i \neq j. \tag{C.1}$$

Being $\iota(\cdot)$ a linear operator, $\iota(A + B) = \iota(A) + \iota(B),\ \forall A, B \in \mathbb{R}^{n\times m}$. Furthermore, given $A \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times p}$, with $R = \iota(A\iota(C))$ and $Q = \iota(AC)$, we have

$$r_{ii} = \sum_{j=1}^{m} a_{ij} \left(\sum_{k=1}^{p} c_{jk} \right) = \sum_{k=1}^{p} \sum_{j=1}^{m} a_{ij} c_{jk} = q_{ii}, \quad i = 1 \dots n, \tag{C.2}$$

and $\iota(A\iota(C)) = \iota(AC)$. Notice that, using this linear operator, the zonotopic inclusion of the product of an interval matrix $\mathbf{J} \in \mathbb{I}^{n\times m}$ and a centered zonotope $X = M\mathcal{B}_\infty^{n_g} \subseteq \mathbb{R}^m$ can be compactly represented by:

$$\diamond(\mathbf{J}X) = \begin{pmatrix} mid(\mathbf{J})M & \iota(rad(\mathbf{J})|M|) \end{pmatrix} \mathcal{B}_\infty^{n_g+n}, \tag{C.3}$$

and the interval hull of X is given by $\mathbb{I}(X) = \iota(|M|)\mathcal{B}_\infty^{m}$. Based on these definitions, Theorem C.1 presents some useful properties of the zonotopic inclusion.

Theorem C.1. *Given $X = M\mathcal{B}_\infty^{n_{g1}}, Y = N\mathcal{B}_\infty^{n_{g2}} \subseteq \mathbb{R}^m$ centered zonotopes, and $\mathbf{A}, \mathbf{B} \in \mathbb{I}^{n\times m}$ interval matrices, we have*

(i) $\diamond(\mathbf{A}(X \oplus Y)) = \diamond(\mathbf{A}X) \oplus \diamond(\mathbf{A}Y)$,

(ii) $\diamond((\mathbf{A} + \mathbf{B})X) \subseteq \diamond(\mathbf{A}X) \oplus \diamond(\mathbf{B}X)$,

(iii) *If $X \subseteq Y$, then* $\diamond(\mathbf{A}X) \subseteq \diamond(\mathbf{A}Y)$.

Proof. (i) From (C.2) and the fact that, given $P_1, P_2 \in \mathbb{R}^{n\times n}$ positive diagonal matrices, $(P_1 + P_2)\mathcal{B}^n_\infty = P_1\mathcal{B}^n_\infty \oplus P_2\mathcal{B}^n_\infty$, we have

$$\begin{aligned}
\diamond(\mathbf{A}(X \oplus Y)) &= mid(\mathbf{A})(X \oplus Y) \oplus \iota\left(rad(\mathbf{A})\left(|M| \quad |N|\right)\right)\mathcal{B}^n_\infty \\
&= mid(\mathbf{A})X \oplus mid(\mathbf{A})Y \oplus \iota(rad(\mathbf{A})(\iota(|M|) + \iota(|N|)))\mathcal{B}^n_\infty \\
&= mid(\mathbf{A})X \oplus mid(\mathbf{A})Y \oplus (\iota(rad(\mathbf{A})|M|) + \iota(rad(\mathbf{A})|N|))\mathcal{B}^n_\infty \\
&= (mid(\mathbf{A})X \oplus \iota(rad(\mathbf{A})|M|)\mathcal{B}^n_\infty) \oplus (mid(\mathbf{A})Y \oplus \iota(rad(\mathbf{A})|N|)\mathcal{B}^n_\infty) \\
&= \diamond(\mathbf{A}X) \oplus \diamond(\mathbf{A}Y).
\end{aligned}$$

(ii) First, notice that we have

$$\begin{aligned}
\diamond((\mathbf{A}+\mathbf{B})X) &= (mid(\mathbf{A}) + mid(\mathbf{B}))X \oplus \iota((rad(\mathbf{A}) + rad(\mathbf{B}))|M|)\mathcal{B}^n_\infty, \\
\diamond(\mathbf{A}X) \oplus \diamond(\mathbf{B}X) &= mid(\mathbf{A})X \oplus \iota(rad(\mathbf{A})|M|)\mathcal{B}^n_\infty \oplus mid(\mathbf{B})X \oplus \iota(rad(\mathbf{B})|M|)\mathcal{B}^n_\infty \\
&= mid(\mathbf{A})X \oplus mid(\mathbf{B})X \oplus \iota((rad(\mathbf{A}) + rad(\mathbf{A}))|M|)\mathcal{B}^n_\infty
\end{aligned}$$

and therefore, since $(mid(\mathbf{A}) + mid(\mathbf{B}))X \subseteq mid(\mathbf{A})X \oplus mid(\mathbf{B})X$, we have $\diamond((\mathbf{A}+\mathbf{B})X) \subseteq \diamond(\mathbf{A}X) \oplus \diamond(\mathbf{B}X)$.

(iii) From $X \subseteq Y$, we have $mid(\mathbf{A})X \subseteq mid(\mathbf{A})Y$. It is thus sufficient to prove that $\iota(rad(\mathbf{A})|M|)\mathcal{B}^n_\infty \subseteq \iota(rad(\mathbf{A})|N|)\mathcal{B}^n_\infty$. Since $rad(\mathbf{A})X \subseteq rad(\mathbf{A})Y$, the interval hull of the zonotope $rad(\mathbf{A})X$ is contained in the interval hull of $rad(\mathbf{A})Y$. $\iota(rad(\mathbf{A})|M|)\mathcal{B}^n_\infty \subseteq \iota(rad(\mathbf{A})|N|)\mathcal{B}^n_\infty$ then follows from the fact that $rad(\mathbf{A}) \geq 0$.

□